ON SYMMETRIC BASIC SEQUENCES IN LORENTZ SEQUENCE SPACES II

BY

P. G. CASAZZA AND BOR-LUH LIN

ABSTRACT

It is shown that if $\{y_n\}$ is a block of type I of a symmetric basis $\{x_n\}$ in a Banach space X, then $\{y_n\}$ is equivalent to $\{x_n\}$ if and only if the closed linear span $[y_n]$ of $\{y_n\}$ is complemented in X. The result is used to study the symmetric basic sequences of the dual space of a Lorentz sequence space $d(a,p)$. Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, for $1 \leq p < +\infty$. It is shown that every infinite-dimensional subspace of $d(a,p)$ (respectively, $[f_n]$) has a complemented subspace isomorphic to l_p (respectively, l_q, $1/p + 1/q = 1$ when $1 < p < +\infty$ and c_0 when $p = 1$) and numerous other results on complemented subspaces of $d(a,p)$ and $[f_n]$ are obtained. We also obtain necessary and sufficient conditions such that $[f_n]$ have exactly two non-equivalent symmetric basic sequences. Finally, we exhibit a Banach space X with symmetric basis $\{x_n\}$ such that every symmetric block basic sequence of $\{x_n\}$ spans a complemented subspace in X but X is not isomorphic to either c_0 or l_p, $1 \leq p < +\infty$.

Let $1 \leqq p < +\infty$. For any $a = (a_1, a_2, \cdots) \in c_0 \setminus l_1$, $a_1 \geqq a_2 \geqq \cdots \geqq 0$, let $d(a,p) = \{x = (\alpha_1, \alpha_2, \cdots) \in c_0 : \sup_{\sigma \in \pi} \sum_{i=1}^{\infty} |\alpha_{\sigma(n)}|^p a_n < +\infty\}$ where π is the set of all permutations of the natural numbers N. Then $d(a,p)$ with the norm $\|x\| = (\sup_{\sigma \in \pi} \sum_{n=1}^{\infty} |\alpha_{\sigma(n)}|^p a_n)^{1/p}$ for $x \in d(a,p)$ is a Banach space and the sequence of unit vector $\{x_n\}$ is a symmetric basis of $d(a,p)$ [4], [5]. Let $\{f_n\}$ be the sequence of biorthogonal functionals of $\{x_n\}$ in $d(a,p)^*$. In this paper, we study the symmetric basic sequences in $[f_n]$, the closed linear span of $\{f_n\}$ in $d(a,p)^*$. For the basic properties of $d(a,p)$ we refer to [4], [5]. In particular, it is known that $d(a,p)$ is reflexive for every $a \in c_0 \setminus l_1$ when $1 < p < +\infty$ [5]. For the results on symmetric basic sequences in $d(a,p)$ we refer the reader to [1]. Another important class of Banach spaces with symmetric bases are the Orlicz sequence spaces which have been studied by J. Lindenstrauss and L. Tzafriri [7], [8], [9].

A basis $\{x_n\}$ of a Banach space X is called symmetric if every permutation

Received April 26, 1973 and in revised form September 20, 1973

$\{x_{\sigma(n)}\}$ of $\{x_n\}$ is a basis of X, equivalent to the basis $\{x_n\}$. Let $\{x_n\}$ be a symmetric basis in a Banach space X. Define

$$|||x||| = \sup_{\sigma \in \pi} \sup_{\substack{|\beta_i| \leq 1 \\ 1 \leq n < +\infty}} \left\| \sum_{i=1}^{n} \beta_i f_i(x) x_{\sigma(i)} \right\|, \qquad x \in X,$$

where $\{f_n\}$ is the sequence of biorthogonal functionals of $\{x_n\}$ in X^*. Then the symmetric norm $|||x|||$, $x \in X$, is an equivalent norm on X. Throughout this paper we shall assume that every Banach space with symmetric basis is equipped with the symmetric norm. It is clear that if $\{x_n, f_n\}$ is the unit vector basis of $d(a, p)$, then the norms in $d(a, p)$ and, respectively, $[f_n]$ are symmetric norms.

Let $\{x_n\}$ be a symmetric basis of a Banach space X and let $\{y_n\}$ be a block of type I of $\{x_n\}$. We show that $\{y_n\}$ is equivalent to $\{x_n\}$ if and only if $[y_n]$ is complemented in X. If $\{x_n, f_n\}$ is the unit vector basis of $d(a, p)$, $1 \leqq p < +\infty$, it is shown in [1] that every infinite-dimensional subspace of $d(a, p)$ has a subspace isomorphic to l_p. In this paper it is shown that, in fact, every infinite-dimensional subspace of $d(a, p)$ (respectively, $[f_n]$) has a complemented subspace isomorphic to l_p (respectively, to l_q where $1/p + 1/q = 1$ when $1 < p < +\infty$ and c_0 when $p = 1$). We also show that for $1 < p < +\infty$ and $1/p + 1/q = 1$, every block basic sequence $\{g_n\}$ of $\{f_n\}$ which is equivalent to the unit vector basis of l_q spans a complemented subspace of $d(a, p)^*$. We obtain several necessary and sufficient conditions such that $[f_n]$ has exactly two non-equivalent symmetric basic sequences. An interesting consequence of this result is that in every Lorentz sequence space $d(a, 1)$ it is impossible for $d(a, 1)$ and $[f_n]$ to have exactly two non-equivalent symmetric basic sequences simultaneously. It is also shown that no subspace of $d(a, p)^*$ with symmetric basis can be isomorphic to any Lorentz sequence space. Finally, we exhibit a Lorentz sequence space $d(a, 1)$ with the property that every symmetric block basic sequence of $\{f_n\}$ spans a complemented subspace of $[f_n]$ but $[f_n]$ is not isomorphic either to c_0 or l_p, $1 \leqq p < +\infty$. We also exhibit a Banach space X with unconditional basis $\{x_n\}$ such that every bounded block basic sequence of $\{x_n\}$ spans a complemented subspace of X but X is not isomorphic either to c_0 or l_p, $1 \leqq p < +\infty$.

The notation and terminology in this paper are essentially those of I. Singer [11]. If $\{x_n\}$ and $\{y_n\}$ are the respective bases of Banach spaces X and Y we say that $\{x_n\}$ dominates $\{y_n\}$, and write $\{x_n\} > \{y_n\}$, in the case where $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in X implies $\sum_{n=1}^{\infty} \alpha_n y_n$ converges in Y. The basis $\{x_n\}$ is equivalent to the basis $\{y_n\}$, and we write $\{x_n\} \sim \{y_n\}$, if $\{x_n\} > \{y_n\}$ and $\{y_n\} > \{x_n\}$.

1.

In this section, we study the blocks of type I-IV of a symmetric basis in a Banach space.

DEFINITION. Let $\{x_n\}$ be a symmetric basis of a Banach space X. For any $\alpha = \sum_{n=1}^{\infty} \alpha_n x_n \in X$, $\alpha_1 \neq 0$ and any $p_1 < p_2 < \cdots$, let

$$y_n^{(\alpha)} = \sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} x_i, \qquad n = 1,2,\cdots.$$

Then $\{y_n^{(\alpha)}\}$ is a bounded block basic sequence of $\{x_n\}$ in X. We shall call $\{y_n^{(\alpha)}\}$ a block of type I of $\{x_n\}$.

DEFINITION. [Z. Altshuler.] Let $\{x_n\}$ be a symmetric basis of a Banach space X. If $\{N_i\}$ are subsets of the natural numbers N, such that for every i, $\bar{\bar{N}}_i = \bar{\bar{N}}$, $N = \bigcup_{i=1}^{\infty} N_i$ and $N_i \wedge N_j = \emptyset$ for all $i \neq j$, then for any $0 \neq \alpha = \sum_{n=1}^{\infty} \alpha_n x_n \in X$, define $u_i^{(\alpha)} = \sum_{j=1}^{\infty} \alpha_j x_{i,j}$ where for every $i = 1,2,\cdots$, $N_i = \{i,j\}$. It is clear that $\{u_n^{(\alpha)}\}$ is a symmetric basic sequence in X. The sequence $\{u_n^{(\alpha)}\}$ is called a block of type II of $\{x_n\}$.

PROPOSITION 1. [Z. Altshuler.] *Let $\{x_n\}$ be a symmetric basis of a Banach space X and let $\alpha = \sum_{n=1}^{\infty} \alpha_n x_n \in X$ such that $\alpha_1 \neq 0$. Then*

(i) *for every block $\{y_n^{(\alpha)}\}$ of type* I *of $\{x_n\}$, there exists a subsequence $\{y_{n_i}^{(\alpha)}\}$ of $\{y_n^{(\alpha)}\}$ which is equivalent to a block of type* II *of $\{x_n\}$.*

(ii) *every block $\{u_n^{(\alpha)}\}$ of type* II *is equivalent to a block $\{y_n^{(\alpha)}\}$ of type* I.

PROOF. (i) Since $\{x_n\}$ is symmetric, we may assume that

$$\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n \geqq \cdots \geqq 0.$$

Let

$$y_n^{(\alpha)} = \sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} x_i,$$

$n = 1,2,\cdots$. If $\sup_{1 \leqq n < +\infty} (p_{n+1} - p_n) < +\infty$, then $\{y_n^{(\alpha)}\}$ is equivalent to $\{x_n\}$ which is certainly equivalent to a block of type II of $\{x_n\}$. Hence we may assume, by switching to a subsequence if necessary, that $p_n - p_{n-1} < p_{n+1} - p_n$, $n = 1,2,\cdots$. Let $u_n^{(\alpha)} = \sum_{j=1}^{\infty} \alpha_j x_{n,j}$, $n = 1,2,\cdots$ be a block of type II. Choose an increasing sequence $\{n_i\}$ such that

$$\left\| \sum_{j=p_{n_i+1}-p_{n_i}}^{\infty} \alpha_j x_j \right\| < \varepsilon/2^i \text{ and let } z_{n_i} = \sum_{j=p_{n_i}+1}^{p_{n_i+1}} \alpha_{j-p_{n_i}} x_{i,j-p_{n_i}}, \qquad i = 1,2,\cdots.$$

Then $\{z_{n_i}\}$ is equivalent to $\{y_{n_i}^{(\alpha)}\}$ and

$$\sum_{i=1}^{\infty} \left\| u_i^{(\alpha)} - z_{n_i} \right\| \leqq \sum_{i=1}^{\infty} \left\| \sum_{j=p_{n_i+1}-p_{n_i}}^{\infty} \alpha_j x_{ij} \right\| < \varepsilon.$$

By a theorem of C. Bessaga and A. Pelczynski [2], $\{u_i^{(\alpha)}\}$ is equivalent to $\{z_{n_i}\}$. Thus $\{u_i^{(\alpha)}\} \sim \{y_{n_i}^{(\alpha)}\}$.

(ii) If $\{u_n^{(\alpha)}\}$ is a block of type II, by the same construction, there exists a block $\{y_n^{(\alpha)}\}$ of type I which is equivalent to $\{u_n^{(\alpha)}\}$. Q.E.D.

COROLLARY 2. *Let $\{x_n\}$ be a symmetric basis of a Banach space X. Then every block of type* I *of $\{x_n\}$ is equivalent to $\{x_n\}$ if and only if every block of type* II *of $\{x_n\}$ is equivalent to $\{x_n\}$.*

PROOF. Let $\alpha = \sum_{n=1}^{\infty} \alpha_n x_n \in X$, $\alpha_1 \neq 0$ and let $\{y_n^{(\alpha)}\}$, respectively $\{u_n^{(\alpha)}\}$, be a block of type I, respectively type II, of $\{x_n\}$ determined by α. Since $\{x_n\}$ is symmetric, $\{u_n^{(\alpha)}\} > \{y_n^{(\alpha)}\} > \{x_n\}$. Hence $\{u_n^{(\alpha)}\} \sim \{x_n\}$ implies that $\{y_n^{(\alpha)}\} \sim \{x_n\}$. Conversely, if $\{y_n^{(\alpha)}\} \sim \{x_n\}$ by all $0 \neq \alpha \in X$, by Proposition 1, we conclude that $\{u_n^{(\alpha)}\} \sim \{x_n\}$. Q.E.D.

PROPOSITION 3. *Let $\{x_n\}$ be a symmetric basis of a Banach space X and let*

$$y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} x_i \text{ and } z_n = \sum_{i=p_n+1}^{p_{n+1}} \beta_{i-p_n} x_i, \qquad n = 1,2,\cdots$$

where $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$ and $\beta_1 \geqq \beta_2 \geqq \cdots \geqq 0$, be blocks of type I *in X. If there exists constant $K > 0$ such that*

$$\sum_{i=1}^{n} \alpha_i \leqq K \sum_{i=1}^{n} \beta_i, \qquad n = 1,2,\cdots,$$

then $\{z_n\}$ dominates $\{y_n\}$. A similar result also holds when $\{y_n\}$ and $\{z_n\}$ are blocks of type II.

PROOF. Suppose $\sum_{n=1}^{\infty} b_n z_n$ is convergent. Since $\{x_n\}$ is symmetric, we may assume that $b_n \geqq 0$, $n = 1,2,\cdots$. Let $f \in X^*$, $\|f\| = 1$ and let $f(x_n) = a_n \geqq 0$ $n = 1,2,\cdots$. For each n, let σ_n be a permutation of $\{p_n + 1,\cdots, p_{n+1}\}$ such that $a_{\sigma_n(p_n+1)} \geqq \cdots \geqq a_{\sigma_n(p_{n+1})}$. Then, since $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$,

$$\left| f\left(\sum_{n=1}^{\infty} b_n y_n \right) \right| = \sum_{n=1}^{\infty} b_n \sum_{i=p_n+1}^{p_{n+1}} a_i \alpha_{i-p_n} \leqq \sum_{n=1}^{\infty} b_n \sum_{i=p_n+1}^{p_{n+1}} a_{\sigma_n(i)} \alpha_{i-p_n}.$$

Since

$$\sum_{i=1}^{n} \alpha_i \leqq K \sum_{i=1}^{n} \beta_i, \qquad n = 1,2,\cdots,$$

$$\sum_{i=p_n+1}^{p_n+1} \alpha_{\sigma(i)}\alpha_{i-p_n} \leqq K \sum_{i=p_n+1}^{p_n+1} a_{\sigma_n(i)}\beta_{i-p_n}, \qquad n = 1,2,\cdots.$$

Define $g(x_i) = a_{\sigma_n(i)}$ if $p_n + 1 \leqq i \leqq p_{n+1}$ and extend g linearly to X. Then, since $\{x_n\}$ is symmetric, $\| g \| = \|f\| = 1$ and

$$\left|f\left(\sum_{n=1}^{\infty} b_n y_n\right)\right| \leqq Kg\left(\sum_{n=1}^{\infty} b_n z_n\right) \leqq K \left\|\sum_{n=1}^{\infty} b_n z_n\right\|.$$

Thus

$$\left\|\sum_{n=1}^{\infty} b_n y_n\right\| \leqq K \left\|\sum_{n=1}^{\infty} b_n z_n\right\|. \qquad \text{Q.E.D.}$$

LEMMA 4. *Let $\{x_n\}$ be a symmetric basis of a Banach space X. Then the following statements are equivalent.*

(i) *Every block of type I of $\{x_n\}$ is equivalent to $\{x_n\}$.*

(ii) *For any $\sum_{n=1}^{\infty} \alpha_n x_n \in X$, $\sum_{n=1}^{\infty} \beta_n x_n$ is convergent in X where $\{\beta_n\}$ is any enumeration of the double sequence $\{\alpha_i\alpha_j\}$, $i,j = 1,2,\cdots$.*

(iii) *There exists a constant $K > 0$ such that for any*

$$\sum_{n=1}^{\infty} \alpha_n x_n \in X, \quad \left\|\sum_{n=1}^{\infty} \beta_n x_n\right\| \leqq K \left\|\sum_{n=1}^{\infty} \alpha_n x_n\right\|^2$$

where $\{\beta_n\}$ is any enumeration of $\{\alpha_i\alpha_j\}$, $i,j = 1,2,\cdots$.

PROOF. (i) $\Rightarrow$ (ii). Let $\sum_{n=1}^{\infty}\alpha_n x_n \in X$, $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$ and let

$$y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} x_i, \qquad n = 1,2,\cdots$$

where $p_{n+1} - p_n > p_n - p_{n-1}$, $n = 2,3,\cdots$. Since $\{y_n\} \sim \{x_n\}$, there exists a constant K such that $\sup_{1 \leqq n < +\infty} \| \sum_{i=1}^{\infty} \alpha_i y_{n+i} \| \leqq K$. Let $\{b_n\}$ be any enumeration of $\{\alpha_i\alpha_j\}$, $i,j = 1,2,\cdots$. For a fixed n, there exists n' such that $b_k \in \{\alpha_i\alpha_j\}$, $i,j = 1,2,\cdots,n'$ for all $k = 1,2,\cdots,n$. Choose m such that $p_{m+1} - p_m \geqq n'$. Then $\| \sum_{i=1}^{n} b_i x_i \| \leqq \| \sum_{i=1}^{n'} \alpha_i y_{m+i} \| \leqq K$. Thus $\sum_{n=1}^{\infty} b_n x_n$ is convergent in X.

(ii) $\Rightarrow$ (iii). Let $\{N_i\}$, $i = 1,2,\cdots$ be subsets of the natural numbers such that $N = \bigcup_{i=1}^{\infty} N_i$, $\overline{\overline{N_i}} = \overline{\overline{N}}$, $i = 1,2,\cdots$, and $N_i \wedge N_j = \varnothing$ for all $i \neq j$. Let $N_i = \{(i,j): j = 1,2,\cdots\}$.

For each $0 \neq x = \sum_{n=1}^{\infty} \alpha_n x_n \in X$, let $y_j = \sum_{j=1}^{\infty} \alpha_i x_{i,j}$, $j = 1,2,\cdots$. Then $\{y_j\}$ is a bounded block of type II of $\{x_n\}$ and thus is a basic sequence. For any

$\sum_{n=1}^{\infty}\gamma_n x_n \in X$, since $\sum_{n=1}^{\infty}(|\gamma_n| + |\alpha_n|)x_n$ is convergent, $\sum_{n=1}^{\infty}\beta_n x_n$ converges in X where $\{\beta_n\}$ is an enumeration of $\{(|\gamma_i| + |\alpha_i|)\,(|\gamma_j| + |\alpha_j|)\}$, $i,j = 1,2\cdots$. Thus $\sum_{j=1}^{\infty}\gamma_j\sum_{i=1}^{\infty}\alpha_i x_{i,j}$ converges in X. Define $T_x(\sum_{n=1}^{\infty}\gamma_n x_n) = \sum_{n=1}^{\infty}\gamma_n y_n$ for all $\sum_{n=1}^{\infty}\gamma_n x_n \in X$. Then T_x is a bounded linear operator on X for each $x \in X$. Now for each $y \in X$, $\sup_{\|x\|=1}\| T_x(y)\| = \sup_{\|x\|=1}\| T_y(x)\| = \| T_y \| < +\infty$. By the uniform boundedness principle, there exists a constant $K > 0$ such that $\| T_x \| \leqq K$ for all $\| x \| = 1$ in X. Therefore for any $x = \sum_{n=1}^{\infty}\alpha_n x_n \in X$, $\| \sum_{n=1}^{\infty}\beta_n x_n \| = \| T_x(x)\| = \| T_{x/\|x\|}(x)\| \cdot \| x \| \leqq K\| x \|^2 = K\| \sum_{n=1}^{\infty}\alpha_n x_n \|^2$ where $\{\beta_n\}$ is an enumeration of $\{\alpha_i\alpha_j\}$, $i,j = 1,2,\cdots$.

(iii) $\Rightarrow$ (i). Let $\sum_{n=1}^{\infty}\alpha_n x_n \in X$, $\alpha_1 \neq 0$, and let

$$y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} x_i, \qquad n = 1,2,\cdots,$$

be a block of type I. We may assume that $\alpha_n \geqq 0$, $n = 1,2,\cdots$. Since $\alpha_1 \neq 0$, so $\{y_n\} > \{x_n\}$. Conversely, if $\sum_{n=1}^{\infty} b_n x_n$ converges in X and $b_n \geqq 0$, $n = 1,2,\cdots$, then $\sum_{n=1}^{\infty}(\alpha_n + b_n)x_n$ is convergent in X. Thus $\sum_{n=1}^{\infty}\beta_n x_n$ is convergent where $\{\beta_n\}$ is any enumeration of $\{(\alpha_i + b_i)\,(\alpha_j + b_j)\}$, $i,j = 1,2,\cdots$. Now

$$\sup_{1\leqq n<+\infty}\left\| \sum_{i=1}^{n} b_i y_i \right\| \leqq \sup_{1\leqq n<+\infty}\left\| \sum_{i=1}^{n} \beta_i x_i \right\| = \left\| \sum_{n=1}^{\infty} \beta_n x_n \right\|.$$

Thus $\sum_{n=1}^{\infty} b_n y_n$ is convergent in X and so $\{y_n\}$ is equivalent to $\{x_n\}$. Q.E.D.

Definition. Let $\{x_n, f_n\}$ be a symmetric basis of a Banach space X and let $f \in X^*$ such that $f(x_1) \neq 0$. A block of type III of $\{f_n\}$ is a block basic sequence $\{g_n\}$ of $\{f_n\}$ of the form $g_n = \sum_{i=p_n+1}^{p_{n+1}} f(x_{i-p_n})f_i$, $n = 1,2,\cdots$, where $\{p_n\}$ is a strictly increasing sequence of natural numbers. If $\{N_i\}$ is a sequence of subsets of N such that $N = \bigcup_{i=1}^{\infty} N_i$, then $N_i \wedge N_j = \varnothing$ for all $i \neq j$ and $\bar{\bar{N}}_i = \bar{\bar{N}}$, $i = 1,2,\cdots$. Define $g_i(\sum_{j=1}^{\infty}\beta_j x_j) = \sum_{j=1}^{\infty} f(x_j)\beta_{i,j}$ for all $\sum_{j=1}^{\infty}\beta_j x_j \in X$ where

$$N_i = \{(i,j): j = 1,2,\cdots\}, \qquad i = 1,2,\cdots.$$

We shall call $\{g_i\}$ a block of type IV of $\{f_n\}$.

Since $\{x_n\}$ is symmetric, it is easy to see that $\| g_i \| = \| f \|$, $i = 1,2,\cdots$, and $\| \sum_{i=1}^{n} b_i g_i \| \leqq \| \sum_{i=1}^{n+m} b_i g_i \|$ for any $b_1, b_2, \cdots, b_{n+m}$, $n, m = 1,2,\cdots$. Hence $\{g_n\}$ is a bounded basic sequence in X^*.

The proof of the following proposition is straightforward and is omitted.

Proposition 5. *Let $f \in X^*$, $f(x_1) \neq 0$ and let $\{g_n\}$, respectively $\{h_n\}$, be a block of type* III, *respectively type* IV, *of $\{f_n\}$ determined by f. Then $\{h_n\} > \{g_n\} > \{f_n\}$*

LEMMA 6. *Let $\{x_n, f_n\}$ be a symmetric basis of a Banach space X. Then the following statements are equivalent.*

(i) *Every block of type* I *of $\{f_n\}$ is equivalent to $\{f_n\}$.*

(ii) *Every block of type* II *of $\{f_n\}$ is equivalent to $\{f_n\}$.*

(iii) *Every block of type* III *of $\{f_n\}$ is equivalent to $\{f_n\}$.*

(iv) *Every block of type* IV *of $\{f_n\}$ is equivalent to $\{f_n\}$.*

PROOF. It is clear that every block of type I, respectively type II, is a block of type III, respectively type IV. By Corollary 2 and Proposition 5, it remains to show that (ii) implies (iv). Let $f \in X^*$, $f(x_1) \neq 0$ and

$$g_i\left(\sum_{j=1}^{\infty} \beta_j x_j\right) = \sum_{j=1}^{\infty} f(x_j)\beta_{i,j}$$

where $\sum_{j=1}^{\infty} \beta_j x_j \in X$, $n = 1, 2, \cdots$. It is clear that $\{g_n\} > \{f_n\}$. Conversely, suppose $f = \sum_{n=1}^{\infty} \alpha_n f_n$ is convergent in X^*. Let $h_i = \sum_{j=1}^{\infty} \alpha_j f_{i,j}$ and $h_i' = \sum_{j=1}^{\infty} \alpha_j f_{j,i}$, $i = 1, 2, \cdots$. Since $\{f_n\}$ is symmetric, $\{h_i'\} \sim \{h_i\}$ and by (ii), $\{h_n\} \sim \{f_n\}$. Let $K > 0$ be a constant such that $\| \sum_{n=1}^{\infty} b_n h_n' \| \leqq K \| \sum_{n=1}^{\infty} b_n f_n \|$ for all $\sum_{n=1} b_n f_n \in X^*$. Now for any $\sum_{n=1}^{\infty} \beta_n x_n \in X$ and any $m = 1, 2, \cdots$,

$$\left(\sum_{i=1}^{n} \alpha_i g_i\right) \left(\sum_{j=1}^{m} \beta_i x_j\right) = \left(\sum_{j=1}^{m} f(x_j) \sum_{i=1}^{n} \alpha_i f_{i,j}\right) \left(\sum_{k=1}^{\infty} \beta_k x_k\right).$$

Hence

$$\left\|\sum_{i=1}^{n} \alpha_i g_i\right\| \leqq \sup_{1 \leqq m < +\infty} \left\|\sum_{j=1}^{m} f(x_j) \sum_{i=1}^{n} \alpha_i f_{i,j}\right\| \leqq \sup_{1 \leqq m < +\infty} \left\|\sum_{j=1}^{m} f(x_j) h_j' \right\|$$

$$\leqq K \sup_{1 \leqq m < +\infty} \left\|\sum_{j=1}^{m} f(x_j) f_j\right\| \leqq K\|f\|, \qquad n = 1, 2, \cdots.$$

Thus $\sum_{n=1}^{\infty} \alpha_n g_n$ is convergent and therefore $\{g_n\} \sim \{f_n\}$. Q.E.D.

The following lemma is due to J. Lindenstrauss and T. Tzafriri. (See the proof of [6, Th. 4] and also [13].)

LEMMA 7. *Let $\{x_n\}$ be an unconditional basis of a Banach space X. Let $\{\eta_n\} \in c_0$ and $y_n = \sum_{i \in \sigma_n} \alpha_i x_i$, $z_n = \sum_{j \in \tau_n} \beta_j x_j$, $n = 1, 2, \cdots$, be bounded block basic sequences of $\{x_n\}$ such that $\sigma_n \wedge \tau_m = \emptyset$ for all $n, m = 1, 2, \cdots$. If there exists a projection P from X onto $[\eta_n y_n + z_n]$ then $\{z_n\}$ dominates $\{\eta_n y_n\}$.*

PROOF. We may assume that the unconditional constant of $\{x_n\}$ is 1. Suppose $P(y_i) = \sum_{j=1}^{\infty} c_j^{(i)}(\eta_j y_j + z_j)$ and $P(z_i) = \sum_{j=1}^{\infty} d_j^{(i)}(\eta_j y_j + z_j)$, $i = 1, 2, \cdots$. Since $\{x_n\}$ is unconditional, there exists a projection E of norm one such that $E(x_n) = x_n$ if $n \in \sigma_i$ for some $i = 1, 2, \cdots$ and $E(x_n) = 0$ otherwise. Then

$$EP(z_i) = \sum_{j=1}^{\infty} d_j^{(i)} \eta_j y_j$$

and EP can be regarded as an operator from $[z_n]$ to $[y_n]$ which is defined by the infinite matrix $(d_j^{(i)}\eta_j)$. Since $\{z_n\}$ and $\{y_n\}$ are unconditional bases of constant one, it follows that the diagonal matrix defines an operator D: $[z_n] \to [y_n]^{**}$ [12] such that $\| D \| \leqq \| EP \| \leqq \| P \|$.

Suppose that $\sum_{n=1}^{\infty} a_n z_n$ converges. Then $D(\sum_{n=1}^{\infty} a_n z_n) = \sum_{n=1}^{\infty} a_n d_n^{(n)} \eta_n y_n$ converges. However, $\eta_n c_n^{(n)} + d_n^{(n)} = 1$ for all $n = 1, 2, \cdots$, $|c_n^{(n)}| \leqq \| P \|$ and $\lim_{n\to\infty} \eta_n = 0$. Hence $\lim_{n\to\infty} d_n^{(n)} = 1$ and thus $\sum_{n=1}^{\infty} a_n \eta_n y_n$ converges. This completes the proof that $\{z_n\}$ dominates $\{\eta_n y_n\}$. Q.E.D.

We now prove the main theorem of the section.

THEOREM 8. *Let $\{x_n\}$ be a symmetric basis in a Banach space X. If $\{y_n\}$ is a block of type* I *of $\{x_n\}$ then $\{y_n\}$ is equivalent to $\{x_n\}$ if and only if $[y_n]$ is complemented in X.*

PROOF. Let $\sum_{n=1}^{\infty} \alpha_n x_n \in X$, $\alpha_1 \neq 0$, and let $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} x_i$, $n = 1, 2, \cdots$. We may assume that $1 \geqq \alpha_i \geqq 0$, $i = 1, 2, \cdots$. Since $\{x_n\}$ is symmetric we have $\| \sum_{n=1}^{\infty} \beta_n x_n \| = \| \sum_{n=1}^{\infty} \beta_n x_{p_n+1} \|$ for all $\sum_{n=1}^{\infty} \beta_n x_n \in X$. Suppose $\{y_n\}$ is equivalent to $\{x_n\}$. Let $K > 0$ be a constant such that

$$\left\| \sum_{n=1}^{\infty} \beta_n y_n \right\| \leqq K \left\| \sum_{n=1}^{\infty} \beta_n x_n \right\| \text{ for all } \sum_{n=1}^{\infty} \beta_n x_n \in X.$$

Define

$$P\left(\sum_{n=1}^{\infty} \beta_n x_n\right) = \sum_{n=1}^{\infty} (\beta_{p_n+1}/\alpha_1) y_n, \quad \sum_{n=1}^{\infty} \beta_n x_n \in X.$$

Since $\{y_n\}$ is equivalent to $\{x_n\}$, P is well defined and it is easy to see that P is a projection on $[y_n]$ with $\| P \| \leqq K/\alpha_1$.

Conversely, let P be a projection from X onto $[y_n]$. If $x = \sum_{n=1}^{\infty} \beta_n x_n$ in X and $\| x \| \leqq 1$, choose $1 = n_1 < n_2 < \cdots$ such that $\| \sum_{j=n_i}^{\infty} \beta_j x_j \| \leqq 1/2^i$, $i = 2, 3, \cdots$. For $n_i \leqq m < n_{i+1}$, $i = 1, 2, \cdots$, let

$$z_m = \begin{cases} \sum_{j=1}^{i} \alpha_j x_{p_m+j} & \text{if } p_m + i \leqq p_{m+1} \\ y_m & \text{if } p_m + i > p_{m+1} \end{cases}$$

and

$$w_m = \begin{cases} (y_m - z_m)/\| y_m - z_m \| & \text{if } y_m \neq z_m \\ 0 & \text{if } y_m = z_m. \end{cases}$$

Let $\eta_n = \| y_n - z_n \|$, $n = 1,2,\cdots$. Then $y_n = \eta_n w_n + z_n$, $n = 1,2,\cdots$, and $\{z_n\}$ and $\{w_n\}$ are bounded blocks of $\{x_n\}$, $\{\eta_n\} \in c_0$. Since $0 \leqq \alpha_n \leqq 1$, $n = 1,2,\cdots$, $\| \sum_{n=1}^{\infty} \beta_n z_n \| \leqq \sum_{i=1}^{\infty} \alpha_i \| \sum_{j=n_i}^{\infty} \beta_j x_{p_j+i} \| < +\infty$. Thus $\sum_{n=1}^{\infty} \beta_n z_n$ is convergent. By Lemma 4, $\sum_{n=1}^{\infty} \beta_n \eta_n w_n$ is convergent and so $\sum_{n=1}^{\infty} \beta_n y_n$ is convergent. Hence $\{x_n\} > \{y_n\}$. However, $\{y_n\} > \{x_n\}$. This completes the proof that $\{y_n\} \sim \{x_n\}$.

REMARK 9. The projection constructed in the proof of Theorem 8 can be constructed in a much more general setting. In particular, if $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$ is a normalized block basic sequence of a symmetric basis $\{x_n\}$ of a Banach space X which is equivalent to $\{x_n\}$ and $\inf_{1 \leqq n} \sup_{p_n+1 \leqq i \leqq p_{n+1}} |\alpha_i| \geqq c > 0$, then there exists a projection P of X onto $[y_n]$. We have only to define

$$P\left(\sum_{n=1}^{\infty} b_n x_n\right) = \sum_{n=1}^{\infty} (b_{i_n}/a_{i_n}) y_n$$

where $p_n + 1 \leqq i_n \leqq p_{n+1}$ have been chosen to satisfy $|\alpha_{i_n}| \geqq c$ for $n = 1,2,\cdots$.

REMARK 10. By similar argument, it can be proved that Theorem 8 also holds for blocks of type II and type III. By using Theorem 8, it is easy to construct a Banach space X with symmetric basis $\{x_n\}$ such that there exist symmetric block basic sequences of $\{x_n\}$ which span a non-complemented subspace in X.

COROLLARY 11. *Let $\{x_n\}$ be the unit vector basis of $d(a,p)$, $1 \leqq p < +\infty$. Then $d(a,p)$ has exactly two non-equivalent symmetric basic sequences if and only if every block of type* I *of $\{x_n\}$ spans a complemented subspace of $d(a,p)$.*

PROOF. If $d(a,p)$ has exactly two non-equivalent symmetric basic sequences, then every block of type II of $\{x_n\}$ is equivalent to $\{x_n\}$. By Corollary 2, every block of type I of $\{x_n\}$ is equivalent to $\{x_n\}$ and hence spans a complemented subspace in $d(a,p)$. Conversely, if every block of type I of $\{x_n\}$ spans a complemented subspace in $d(a,p)$ then every block of type I of $\{x_n\}$ is equivalent to $\{x_n\}$. Hence $d(a,p)$ has exactly two non-equivalent symmetric basic sequences [1, Cor. 5]. Q.E.D.

COROLLARY 12. *Let X be a subspace of $d(a,p)$, $1 \leqq p < +\infty$. If X is isomorphic to $d(a,p)$ then there exists a complemented subspace Y of X in $d(a,p)$ such that Y is isomorphic to $d(a,p)$.*

PROOF. Let $\{x_n\}$ be the unit vector basis of $d(a,p)$ and let $\{u_n\}$ be a basis in X which is equivalent to $\{x_n\}$. Then there exists a bounded block basic sequence $\{y_n\}$ of $\{x_n\}$ and a subsequence $\{k_n\}$ of the integers such that $\sum_{n=1}^{\infty} \| y_n - u_{k_n} \| < 1$.

Let $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$, $n = 1,2,\cdots$, and let $\{g_n\}$ be the associated sequence of biorthogonal functionals of $\{y_n\}$. Since $\{y_n\}$ is equivalent to $\{x_n\}$, by [1, Lem. 1], there exists a constant $c > 0$ such that $\inf_{1 \leqq n} \sup_{p_n+1 \leqq i \leqq p_{n+1}} |\alpha_i| \geqq c > 0$. By Remark 9, there exists a projection P from X onto $[y_n]$. Choose $m \in N$ such that $\| E_m P \| \sum_{n=m}^{\infty} \| g_n \| \| y_n - u_{k_n} \| < 1$ where E_m is the projection on $[y_n]$ defined by $E_m(y_n) = y_n$ for $n \geqq m$ and $E_m(y_n) = 0$ otherwise. By [2, Th. 2], $Y = [u_{k_n}]$, $n = m, m+1, \cdots$, is complemented in $d(a,p)$. It is clear that Y is isomorphic to $d(a,p)$. Q.E.D.

PROPOSITION 13. *Let $\{x_n\}$ be a symmetric basis of a Banach space X and let $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$, $n = 1,2,\cdots$, be a block basic sequence of $\{x_n\}$. For each n, let σ_n be a permutation of $\{p_n + 1, \cdots, p_{n+1}\}$ and let $z_n = \sum_{i=p_n+1}^{p_{n+1}} |\alpha_{\sigma(i)}| x_i$. Then $[y_n]$ is complemented in X if and only if $[z_n]$ is complemented in X.*

PROOF. Obvious.

COROLLARY 14. *Let Y be a Banach space with symmetric basis $\{y_n\}$. If Y is isomorphic to a complemented subspace of $d(a,p)$, $1 \leqq p < +\infty$, then Y is isomorphic to either l_p or $d(a,p)$. In particular. let $d(a,p)$ and $d(b,p)$ be Lorentz sequence spaces; then $d(b,p)$ is isomorphic to a complemented subspace of $d(a,p)$ if and only if $d(b,p)$ is isomorphic to $d(a,p)$.*

PROOF. Suppose Y is isomorphic to a complemented subspace X of $d(a,p)$ and X is not isomorphic to l_p. Let $\{x_n\}$ be the unit vector basis of $d(a,p)$. Since $\{x_n\}$ and $\{y_n\}$ are symmetric bases and since X is complemented in $d(a,p)$ by Proposition 13, [1, Th. 3], and [2, Th. 2], $\{y_n\}$ is equivalent to a block $\{z_n\}$ of type I of $\{x_n\}$ and we may choose $\{z_n\}$ such that $[z_n]$ is complemented. Hence $\{z_n\} \sim \{x_n\}$ and thus $\{y_n\} \sim \{x_n\}$. Q.E.D.

DEFINITION. Let $\{s_n\}$ and $\{t_n\}$ be two sequences of non-negative numbers. We say that $\{t_n\}$ dominates $\{s_n\}$, denoted by $t_n > s_n$, if there exists a constant $K > 0$ such that $s_n \leqq K t_n$, $n = 1,2,\cdots$. We say that $\{s_n\}$ is equivalent to $\{t_n\}$, and write $s_n \sim t_n$, if $s_n > t_n$ and $t_n > s_n$.

By [1, Lem. 2], $d(a,p)$ is isomorphic to $d(b,p)$ if and only if $s_n \sim t_n$ where $s_n = \sum_{i=1}^{n} a_i$ and $t_n = \sum_{i=1}^{n} b_i$, $n = 1,2,\cdots$. As a consequence, a Lorentz sequence space $d(a,p)$ is isomorphic to a subspace of $d(b,p)$ if and only if there exists $0 \neq \alpha = \sum_{i=1}^{\infty} \alpha_i x_i \in d(a,p)$ such that $\| \alpha \| = 1$ and $s_n^{(\alpha)} \sim t_n$ where

$$s_n^{(\alpha)} = \sum_{i=1}^{\infty} \alpha_i^p (s_{ni} - S_{n(i-1)}).$$

2.

In this section, we study the symmetric basic sequences in a Lorentz sequence space $d(a,p)$ which span complemented subspaces in $d(a,p)$.

LEMMA 15. *Let $\{x_n\}$ be the unit vector basis of $d(a,p)$, $1 \leqq p < +\infty$. If $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$, $n = 1,2,\cdots$, is a bounded block basic sequence of $\{x_n\}$ such that $\lim_{n\to\infty} \alpha_n = 0$ then there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which is equivalent to the unit vector basis of l_p and such that $[y_{n_i}]$ is complemented in $d(a,p)$. Furthermore, if $\{y_n\}$ is normalized, then $\{y_{n_i}\}$ can be chosen in such a way that the projection P from $d(a,p)$ onto $[y_{n_i}]$ has norm as close to one as desired.*

PROOF. We may assume that $\| y_n \| = 1$, $n = 1,2,\cdots$. By taking a subsequence if necessary, and by Proposition 13, we may assume that $\alpha_{p_1+1} \geqq \alpha_{p_1+2} \geqq \cdots \geqq \alpha_n \geqq \cdots \geqq 0$ and $p_{n+2} - p_{n+1} > p_{n+1} - p_n$, $n = 1,2,\cdots$. By [1, Lem. 1], there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $\{y_{n_i}\}$ is equivalent to the unit vector basis of l_p and

$$\sum_{j=p_{n_i}+1}^{p_{n_i+1}} \alpha_j^p a_{j+\tau(i)} \geqq \frac{1}{2^p}, \qquad i = 1,2,\cdots \tag{1}$$

where $\tau(i) = \sum_{k=1}^{i-1} p_{n_k+1} - p_{n_i}$.

For $\sum_{i=1}^{\infty} \beta_i x_i \in d(a,p)$, define $P(\sum_{i=1}^{\infty} \beta_i x_i) = \sum_{i=1}^{\infty} d_i y_{n_i}$ where for $i = 1,2,\cdots$.

$$d_i = \sum_{j=p_{n_i}+1}^{p_{n_i+1}} \beta_j \alpha_j^{p-1} a_{j+\tau(i)} \Big/ \sum_{j=p_{n_i}+1}^{p_{n_i+1}} \alpha_j^p a_{j+\tau(i)}.$$

It is clear P is a projection onto $[y_{n_i}]$. By [1, Prop. 5] and the Hölder inequality,

$$\begin{aligned}
\left\| P\left(\sum_{i=1}^{\infty} \beta_i x_i\right)\right\|^p &\leqq \sum_{i=1}^{\infty} |d_i|^p \leqq 2^{p^2} \sum_{i=1}^{\infty} \left| \sum_{j=p_{n_i}+1}^{p_{n_i+1}} \beta_j \alpha_j^{p-1} a_{j+\tau(i)} \right|^p \\
&\leqq 2^{p^2} \sum_{i=1}^{\infty} \left[\sum_{j=p_{n_i}+1}^{p_{n_i+1}} |\beta_j|^p a_{j+\tau(i)} \right] \left[\sum_{j=p_{n_i}+1}^{p_{n_i+1}} \alpha_j^p a_{j+\tau(i)} \right]^{p-1} \\
&= 2^{p^2} \sum_{i=1}^{\infty} \left[\sum_{j=p_{n_i}+1}^{p_{n_i+1}} |\beta_j|^p a_{j+\tau(i)} \right] \| y_{n_i} \|^{p(p-1)} \\
&\leqq 2^{p^2} \left\| \sum_{i=1}^{\infty} \beta_i x_i \right\|^p.
\end{aligned}$$

Hence P is well defined and continuous.

Finally, it is possible to replace $1/2^p$ in (1) by a constant as close to one as one desired (see the proof of [1, Lem. 1]), hence the subsequence $\{y_{n_i}\}$ can be chosen in such a way that $\| P \|$ is aribtrarily close to one. Q.E.D.

(The authors wish to thank Dr. Z. Altshuler for pointing out the fact that $\{y_{n_i}\}$ can be chosen such that $\| P \|$ is as close to one as one desires and for providing a second, simpler proof of Corollary 16 which he discovered independently.)

COROLLARY 16. *Let $\{x_n\}$ be the unit vector basis of $d(a,p)$, $1 \leqq p < +\infty$. If $\{y_n\}$ is a block basic sequence of $\{x_n\}$ which is equivalent to the unit vector basis of l_p then there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $[y_{n_i}]$ is complemented in $d(a,p)$. Furthermore, if $\{y_n\}$ is normalized then $\{y_{n_i}\}$ can be chosen in such a way that the projection P from $d(a,p)$ to $[y_{n_i}]$ is of norm arbitrarily close to one.*

FIRST PROOF. Since $\{y_n\}$ is not equivalent to a block of type I of $\{x_n\}$, by [1, Th. 3, case 2], there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ and $z_i = \sum_{j=p_{ni}+1}^{k_i} \alpha_j x_j$, $w_i = y_{n_i} - z_i$ such that $\{w_i\}$ is a bounded block basic sequence of $\{x_n\}$

$$\inf_{1 \leqq n < +\infty} \sup_{k_i+1 \leqq j \leqq p_{ni}+1} |\alpha_j| = 0.$$

By Lemma 15, and switching to a subsequence, we may assume that $[w_i]$ is complemented and $\{w_i\}$ is equivalent to the unit vector basis $\{e_i\}$ of l_p. Let P_0 be a projection from $d(a,p)$ onto $[w_i]$ and let E be the projection on $d(a,p)$ defined by $E(x_j) = x_j$ if $k_i + 1 \leqq j \leqq p_{n_i+1}$ for some $i = 1,2,\cdots$ and $E(x_j) = 0$ otherwise. Then $P_0E(y_{n_i}) = w_i$, $i = 1,2,\cdots$. For any $\sum_{i=1}^{\infty} \beta_i x_i \in d(a,p)$, if $P_0E(\sum_{i=1}^{\infty}\beta_i x_i) = \sum_{i=1}^{\infty} d_i w_i$ then define $P(\sum_{i=1}^{\infty}\beta_i x_i) = \sum_{i=1}^{\infty} d_i y_{n_i}$. Since both $\{y_{n_i}\}$ and $\{w_i\}$ are equivalent to $\{e_i\}$, it is easy to show that P is a well-defined, bounded projection from $d(a,p)$ onto $[y_{n_i}]$. Q.E.D.

SECOND PROOF. [Z. Altshuler.] Suppose that $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$, $n = 1,2,\cdots$, $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$ is a bounded block basic sequence of $\{x_n\}$. Since $\{y_n\}$ is equivalent to the unit vector basis $\{e_n^{(p)}\}$ of l_p, there exists a constant $A > 0$ such that for all $k = 1,2,\cdots$,

$$\left\| \sum_{i=1}^{k} y_i \right\|^p > A \left\| \sum_{i=1}^{k} e_i^{(p)} \right\|^p = kA.$$

Hence

$$\sum_{n=1}^{\infty} \sum_{=p_n+1}^{p_{n+1}} \alpha_i^p a_i / k > A. \text{ Since } \sum_{n=1}^{k} \sum_{i=p_n+1}^{p_{n+1}} \alpha_i^p a_i / k$$

is the average of k numbers, this implies that there exists a subsequence $\{n_i\}$ such that $\sum_{j=p_{ni}+1}^{p_{ni+1}} \alpha_j^p a_j > A/2$. Define the projection P from $d(a,p)$ to $[y_{n_i}]$ by $P(\sum_{i=1}^{\infty}\beta_i x_i) = \sum_{k=1}^{\infty} (\sum_{i=p_{nk}+1}^{p_{nk+1}} \beta_i \alpha_i^{p-1} a_{\tau(i)} / \sum_{i=p_{nk}+1}^{p_{nk+1}} \alpha_i^p a_{\tau(i)}) y_{n_k}$, $\sum_{i=1}^{\infty} \beta_i x_i \in d(a,p)$, where $\tau(i) = \sum_{j=1}^{i-1} (p_{nj+1} - p_{nj}) + i$ for $p_{n_k} < i \leqq p_{n_k+1}$. By a computation similar to that of Lemma 15, P has the desired properties. Q.E.D.

COROLLARY 17. *In $d(a,p)$, $1 \leqq p < +\infty$, every infinite dimensional subspace X contains a complemented subspace which is isomorphic to l_p and every basic sequence $\{y_n\}$ which is equivalent to the unit vector basis of l_p has a subsequence $\{y_{n_i}\}$ such that $[y_{n_i}]$ is complemented in $d(a,p)$.*

PROOF. Let $\{x_n\}$ be the unit vector basis of $d(a,p)$. Then there exist a basic sequence $\{u_n\}$ in X and a block basic sequence $\{y_n\}$ of $\{x_n\}$ such that

$$\sum_{n=1}^{\infty} \| u_n - y_n \| < 1.$$

By [1, Cor. 3] there exists a block basic sequence

$$z_n = \sum_{i=p_n+1}^{p_{n+1}} b_i y_i, \qquad n = 1,2,\cdots,$$

such that $\{z_n\}$ is equivalent to the unit vector basis of l_p and, by Corollary 16, we may assume that there exists a projection P from $d(a,p)$ onto $[z_n]$. Let $\{g_n\}$ be the associated sequence of biorthogonal functionals of $\{z_n\}$ and let

$$w_n = \sum_{i=p_n+1}^{p_{n+1}} b_i u_i, \qquad n = 1,2,\cdots.$$

Then

$$\| P \| \sum_{n=1}^{\infty} \| g_n \| \cdot \| w_n - z_n \| \leqq \| P \| \sum_{n=1}^{\infty} \| g_n \| \cdot \sum_{i=p_n+1}^{p_{n+1}} | b_i | \| u_i - y_i \|$$

$$\leqq K \| P \| \sup_{1 \leqq n < +\infty} \| g_n \| \cdot \sum_{n=1}^{\infty} \| u_n - y_n \| < +\infty$$

where K is a constant such that $\sup |b_n| < +\infty$. Since $\{z_n\}$ is unconditional, the projection E_m on $[z_n]$ (defined by $E_m(z_n) = z_n$ if $n \geqq m$ and $E_m(z_n) = 0$ otherwise) have uniformly bounded norms. Hence there exists an $m \in N$ such that $\| E_m P \| \sum_{n=m}^{\infty} \| g_n \| \| w_n - z_n \| < 1$. Then the subspace $[w_n]$, $n = m, m+1, \cdots$, is complemented in $d(a,p)$ and is isomorphic to l_p. Q.E.D.

REMARK 18. In [1, Th. 1], it is shown that every infinite dimensional subspace X of $d(a,p)$, $1 \leqq p < +\infty$, contains a subspace Y which is isomorphic to l_p and if X has symmetric basis then Y can be chosen to be complemented in X.

THEOREM 19. *$d(a,p)$, $1 \leqq p < +\infty$, has exactly two non-equivalent symmetric basic sequences if and only if for every bounded basic sequence $\{y_n\}$ in $d(a,p)$ there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $\{y_{n_i}\}$ is symmetric and $[y_{n_i}]$ is complemented in $d(a,p)$.*

PROOF. Suppose that $d(a,p)$ has exactly two non-equivalent symmetric basic sequences. By taking a subsequence if necessary, we may assume that $\{y_n\}$ is equivalent to a block basic sequence $\{z_n\}$ of $\{x_n\}$, the unit vector basis of $d(a,p)$, such that $\sum_{n=1}^{\infty} \| y_n - z_n \| < 1$. Let $z_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$, $n = 1,2,\cdots$. By Proposition 13, we may assume that $\alpha_{p_n+1} \geqq \cdots \geqq \alpha_{p_{n+1}} \geqq 0$, $n = 1,2,\cdots$. Now, by [1, Th. 3], there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ which is either equivalent to the unit vector basis of l_p, and hence a subsequence which spans a complemented subspace in $d(a,p)$, or there exists a block $\{w_i\}$ of type I of $\{x_n\}$ such that

$$\sum_{i=1}^{\infty} \| z_{n_i} - w_i \| < 1.$$

By hypothesis and Theorem 8, $[w_i]$ is complemented in $d(a,p)$. Hence, by taking a subsequence if necessary, we may assume that $[z_{n_i}]$ is complemented in $d(a,p)$. Thus, by a similar proof of Corollary 17, $\{y_n\}$ has a subsequence which spans a complemented subspace in $d(a,p)$.

Conversely, if every bounded block basic sequence $\{y_n\}$ of $\{x_n\}$ has a subsequence which spans a complemented subspace, then by Theorem 8, every symmetric block of type I is equivalent to $\{x_n\}$ and so $d(a,p)$ has exactly two non-equivalent symmetric basic sequences. Q.E.D.

REMARK 20. Let us recall that $d(a,p)$, $1 \leqq p < +\infty$, has exactly two non-equivalent symmetric basic sequences if and only if $\sup_{1 \leqq n < +\infty} s_{nk}/s_n s_k < +\infty$ [1, Th. 6].

3.

In this section, we study the symmetric basic sequences in the dual space of a Lorentz sequence space.

PROPOSITION 21. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, $1 < p < +\infty$. Then every bounded block basic sequence of $\{f_n\}$ is q-besselian where $1/p + 1/q = 1$.*

PROOF. Let $g_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i f_i$, $\| g_n \| = 1$, $n = 1,2,\cdots$, be a bounded block basic sequence of $\{f_n\}$. For each $n = 1,2,\cdots$, there exists $y_n = \sum_{i=p_n+1}^{p_{n+1}} \beta_i x_i$ such that $\| y_n \| = 1$ and $g_n(y_n) = 1$. Then $\{y_n\}$ is a bounded block basic sequence of $\{x_n\}$ and is p-hilbertian [1, Prop. 5]. Hence for any $\{c_n\} \in l_p$, $\sum_{n=1}^{\infty} c_n y_n$ is convergent. Suppose $\sum_{n=1}^{\infty} b_n g_n$ is convergent in $d(a,p)^*$, then

$$\sum_{n=1}^{\infty} b_n c_n = \left(\sum_{n=1}^{\infty}{}' b_n g_n\right)\left(\sum_{n=1}^{\infty}{}' c_n y_n\right)$$

is convergent. Thus $\{b_n\} \in l_q$ and $\{f_n\}$ is q-besselian. Q.E.D.

We now present a technical result which will be used in proving the important Lemma 23.

PROPOSITION 22. *Let $\{x_n\}$ be the unit vector basis of $d(a,p)$, $1 \leqq p < +\infty$. Given $\varepsilon > 0$ and $m \in N$ there exists $\delta > 0$ such that*

$$\left\|\sum_{n=1}^{\infty} \beta_n x_n\right\|^p \leqq \sum_{n=1}^{\infty} \beta_n^p a_{n+m} + \varepsilon$$

for all $x = \sum_{n=1}^{\infty} \beta_n x_n \in d(a,p)$ with $\delta \geqq \beta_1 \geqq \beta_2 \geqq \cdots \geqq 0$.

PROOF. We may assume that $1 \geqq a_1 \geqq a_2 \geqq \cdots \geqq 0$. Choose $k \in N$ such that $|a_n| \leqq \varepsilon/2m$ for $n \geqq k$. Let $\delta^p = \min(1, \varepsilon/2k)$. For $\sum_{n=1}^{\infty} \beta_n x_n \in d(a,p)$ with $\delta \geqq \beta_1 \geqq \beta_2 \geqq \cdots \geqq 0$, then

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} \beta_n x_n\right\|^p &= \sum_{n=1}^{\infty} \beta_n^p a_{n+m} + \sum_{n=1}^{k} \beta_n^p (a_n - a_{n+m}) + \sum_{n=k+1}^{\infty} \beta_n^p (a_n - a_{n+m}) \\ &\leqq \sum_{n=1}^{\infty} \beta_n^p a_{n+m} + \sum_{n=1}^{k} \delta^p + \sum_{n=k+1}^{\infty} (a_n - a_{n+m}) \\ &\leqq \sum_{n=1}^{\infty} \beta_n^p a_{n+m} + k\left(\frac{\varepsilon}{2k}\right) + \sum_{n=k+1}^{k+m} a_n \\ &\leqq \sum_{n=1}^{\infty} \beta_n^p a_{n+m} + \frac{\varepsilon}{2} + m\left(\frac{\varepsilon}{2m}\right) \\ &= \sum_{n=1}^{\infty} \beta_n^p a_{n+m} + \varepsilon.\end{aligned}$$

Q.E.D.

LEMMA 23. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a, p)$, $1 \leqq p < +\infty$. I $g_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i f_i$, $n = 1, 2, \cdots$, is a bounded block basic sequence of $\{f_n\}$ such that $\lim_{n\to\infty} \alpha_n = 0$, then there exists a subsequence of $\{g_n\}$ which is equivalent to the unit vector basis of c_0 when $p = 1$, respectively to l_q when $1 < p < +\infty$ and $1/p + 1/q = 1$.*

PROOF. Since $\{f_n\}$ is symmetric (and switching to a subsequence, if necessary) we may assume that $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$ and $\|g_n\| = 1$, $n = 1, 2, \cdots$. Since $\lim_{n\to\infty} \alpha_n = 0$, hence $\sup_{1 \leqq n < +\infty} (p_{n+1} - p_n) = +\infty$.

Case 1. $p = 1$. Observe that

$$\sum_{i=1}^{m} \alpha_{p_n+i} = g_n\left(\sum_{i=1}^{m} x_{p_n+i}\right) \leqq \left\|\sum_{i=1}^{m} x_{p_n+i}\right\| = \sum_{i=1}^{m} a_i$$

for all $1 \leqq m \leqq p_{n+1} - p_n$, $n = 1,2,\cdots$. By induction, we shall find a sequence $1 = n_1 < n_2 < \cdots$ with the following property:

$$\sum_{j=1}^{m} \alpha_{p_{n_i}+j} \leqq 2\sum_{j=1}^{m} a_{l_i+j} \text{ for all } 1 \leqq m \leqq p_{n_i+1} - p_{n_i},$$

$$i = 1,2,\cdots, \text{ where } l_1 = 0 \text{ and } l_i = \sum_{j=1}^{i-1} (p_{n_j+1} - p_{n_j}), \qquad i = 2,3,\cdots.$$

Let $1 = n_1$ and suppose $n_1, n_2, \cdots, n_i$ are chosen with the above property. Choose $k > p_{n_i}$ such that $\sum_{j=1}^{k} a_{l_i+j} \geqq \sum_{j=1}^{l_i} a_j$. Since $\lim \alpha_n = 0$, choose $n_{i+1} > n_i$ such that $p_{n_{i+1}+1} - p_{n_{i+1}} \geqq l_i + k$ and $\alpha_{p_{n_{i+1}}+j} \leqq a_{l_i+j}$ for all $1 \leqq j \leqq k$. Now for $1 \leqq m \leqq p_{n_{i+1}} - p_{n_i}$, and either

(i) $1 \leqq m \leqq k$, then $\sum_{j=1}^{m} \alpha_{p_{n_{i+1}}+j} \leqq \sum_{j=1}^{m} a_{l_i+j} \leqq 2\sum_{j=1}^{m} a_{l_i+j}$; or

(ii) $k < m$, then $\sum_{j=1}^{m} \alpha_{p_{n_{i+1}}+j} \leqq \sum_{j=1}^{m} a_j = \sum_{j=1}^{m} (a_j - a_{l_i+j}) + \sum_{j=1}^{m} a_{l_i+j}$

$$\leqq \sum_{j=1}^{l_i} a_j + \sum_{j=1}^{m} a_{l_i+j} \leqq \sum_{j=1}^{k} a_{l_i+j} + \sum_{j=1}^{m} a_{l_i+j} \leqq 2\sum_{j=1}^{m} a_{l_i+j}.$$

Hence n_{i+1} satisfies the required property. To show that $\{g_{n_i}\}$ is equivalent to the unit vector basis of c_0 (since $\{g_{n_i}\}$ is unconditional) it suffices to show that

$$\sup_{1 \leqq m < +\infty} \left\|\sum_{i=1}^{m} g_{n_i}\right\| < +\infty.$$

Let $x = \sum_{n=1}^{\infty} \beta_n x_n \in d(a,1)$. Since $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$, we may assume that $\beta_1 \geqq \beta_2 \geqq \cdots \geqq 0$. Then

$$\left|\left(\sum_{i=1}^{m} g_{n_i}\right)(x)\right| = \sum_{i=1}^{m} \sum_{j=p_{n_i}+1}^{p_{n_{i+1}}} \alpha_j \beta_j \leqq 2\sum_{i=1}^{m} a_{l_i+j-p_n}\beta_j \leqq 2\|x\|, \qquad m = 1,2,\cdots.$$

Hence $\sup_{1 \leqq m < +\infty} \left\|\sum_{i=1}^{m} g_{n_i}\right\| \leqq 2$.

Case 2. $1 < p < +\infty$. By induction, we shall construct a block basic sequence $h_n = \sum_{i=q_n+1}^{q_{n+1}} \gamma_i f_i$, $n = 1,2,\cdots$, such that

(i) $\|h_n\| = 1$, $n = 1,2,\cdots$, and $\{h_n\}$ is equivalent to a subsequence of $\{g_n\}$; and

(ii) if $x=\sum_{i=q_n+1}^{q_{n+1}}\beta_i x_i \in d(a,p)$, $\|x\| \leqq 1$, $\beta_{q_n+1} \geqq \beta_{q_n+2} \geqq \cdots \geqq \beta_{q_{n+1}} \geqq 0$, then $|h_n(x)| \leqq 1/2^n + (1/2^n + \sum_{i=q_n+1}^{q_{n+1}}\beta_i^p a_i)^{1/p}$, $n = 1,2,\cdots$.

Let $q_1 = p_1$ and $h_1 = g_1$. Since $p_1 = 0$,

$$h_1\left(\sum_{i=q_1+1}^{q_2}\beta_i x_i\right) \leqq \left\|\sum_{i=q_1+1}^{q_2}\beta_i x_i\right\| < \tfrac{1}{2} + \left(\tfrac{1}{2} + \sum_{i=q_1+1}^{q_2}\beta_i^p a_i\right)^{1/p}.$$

Suppose that we have constructed $h_1, h_2,\cdots, h_{n-1}$ with the properties (i), (ii). Let $m = q_n$ and $\varepsilon = 1/2^n$ in Proposition 22; then there exists $\delta > 0$ such that $\|\sum_{i=1}^{\infty}\beta_i x_i\|^p \leqq 1/2^n + \sum_{i=1}^{\infty}\beta_i^p a_{i+q_n}$ for all $\|\sum_{i=1}^{\infty}\beta_i x_i\| \leqq 1$ with $\delta \geqq \beta_1 \geqq \beta_2 \geqq \cdots \geqq 0$. Since for each $\varepsilon > 0$, there exists $n(\varepsilon)$ such that $\|(\varepsilon,\varepsilon,\cdots,\varepsilon,0,\cdots)\| > 1$ where the number of epsilons is $n(\varepsilon)$, thus there exists $m \in N$ such that for all $j \geqq m$, $\sup\{\beta_j : \|\sum_{i=1}^{\infty}\beta_i x_i\| \leqq 1,\ \beta_1 \geqq \beta_2 \geqq \cdots \geqq 0\} < \delta$. Now, since $\lim_{n\to\infty}\alpha_n = 0$, choose k such that $p_{k+1} - p_k > m$ and $\sum_{i=p_k+1}^{p_k+m}\alpha_i \leqq 1/2^n$. Let $q_{n+1} = p_{k+1} - p_k + q_n$, $\gamma_{q_n+i} = \alpha_{p_k+i}$, $i = 1,2,\cdots,\ q_{n+1} - q_n$, and let $h_n = \sum_{i=q_n+1}^{q_{n+1}}\gamma_i f_i$. Then $\|h_n\| = \|g_{p_k}\| = 1$. If $x = \sum_{i=q_n+1}^{q_{n+1}}\beta_i x_i$, $\beta_{q_n+1} \geqq \beta_{q_n+2} \geqq \cdots \geqq \beta_{q_{n+1}} \geqq 0$, $\|x\| \leqq 1$, then $\beta_i < \delta$ for $i \geqq m$ and $1 \geqq \beta_{q_n+1}$. Hence

$$|h_n(x)| \leqq \sum_{i=q_n+1}^{q_n+m}\gamma_i + \sum_{i=q_n+m+1}^{q_{n+1}}\beta_i\gamma_i \leqq \sum_{i=p_k+1}^{p_k+m}\alpha_i + \|h_n\|\cdot\left\|\sum_{i=q_n+m+1}^{q_{n+1}}\beta_i x_i\right\|$$

$$\leqq \frac{1}{2^n} + \left\|\sum_{i=q_n+1}^{q_{n+1}-m}\beta_{i+m}x_i\right\| \leqq \frac{1}{2^n} + \left(\frac{1}{2^n} + \sum_{i=q_n+1}^{q_{n+1}-m}\beta_{i+m}^p a_i\right)^{1/p}$$

$$\leqq \frac{1}{2^n} + \left(\frac{1}{2^n} + \sum_{i=q_n+1}^{q_{n+1}}\beta_i^p a_i\right)^{1/p}.$$

Thus h_n satisfies the properties (ii). Note that $\{h_n\}$ is merely a translation of a subsequence of $\{g_n\}$. To show that $\{h_n\}$ is equivalent to the unit vector basis of l_q, since $\{h_n\}$ is q-besselian by Proposition 21, it remains to show that $\sum_{n=1}^{\infty}c_n h_n$ converges for all $\{c_n\} \in l_q$.

Let $x = \sum_{n=1}^{\infty}\beta_n x_n \in d(a,p)$, $\|x\| \leqq 1$. Then

$$\left|\sum_{n=1}^{\infty}c_n h_n(x)\right| \leqq \left(\sum_{n=1}^{\infty}|c_n|^q\right)^{1/q}\left(\sum_{n=1}^{\infty}|h_n(x)|^p\right)^{1/p}.$$

For each n, let σ_n be a permutation of $\{q_n + 1,\cdots, q_{n+1}\}$ such that $|\beta_{\sigma_n(q_n+1)}| \geqq |\beta_{\sigma_n(q_n+2)}| \geqq \cdots \geqq |\beta_{\sigma_n(q_{n+1})}|$. Let $y = \sum_{n=1}^{\infty}\sum_{i=q_n+1}^{q_{n+1}}|\beta_{\sigma_n}(i)|x_i$. Then $\|y\| = \|x\| = 1$ and $|h_n(x)| \leqq \sum_{i=q_n+1}^{q_{n+1}}\gamma_i|\beta_i| \leqq \sum_{i=q_n+1}^{q_{n+1}}\gamma_i|\beta_{\sigma_n}(i)| = h_n(y)$. Hence, by replacing x by y if necessary, we may assume that $\beta_{q_n+1} \geqq \beta_{q_n+2} \geqq \cdots \geqq \beta_{q_n+1} \geqq 0$, $n = 1,2,\cdots$. Now by (ii),

$$\left(\sum_{n=1}^{\infty} |h_n(x)|^p\right)^{1/p} \leqq \left\{\sum_{n=1}^{\infty}\left[\frac{1}{2^n} + \left(\frac{1}{2^n} + \sum_{i=q_n+1}^{q_n+1} \beta_i^p a_i\right)^{1/p}\right]^p\right\}^{1/p}$$

$$\leqq \left[\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)^p\right]^{1/p} + \left(1 + \sum_{n=1}^{\infty} \sum_{i=q_n+1}^{q_{n+1}} \beta_i^p a_i\right)^{1/p}$$

$$\leqq \left[\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)^p\right]^{1/p} + 2^{1/p}.$$

Thus $\sum_{n=1}^{\infty} c_n h_n$ is convergent and the proof of the lemma is complete. Q.E.D.

THEOREM 24. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, $1 \leqq p < +\infty$ Then*

(i) *every infinite-dimensional subspace X of $[f_n]$ contains a complemented subspace Y which is isomorphic to l_q when $1 < p < +\infty$ where $1/p + 1/q = 1$, respectively to c_0 when $p = 1$;*

(ii) *if X is a subspace of $[f_n]$ with symmetric basis then all symmetric bases in X are equivalent.*

PROOF. By an argument similar to that used to prove [1, Th. 1, 4] and Corollary 17.

COROLLARY 25. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, $1 \leqq p < +\infty$. Then*

(i) *$[f_n]$ is not isomorphic to any subspace of $d(b,q)$ for all b, q;*

(ii) *no subspace of $[f_n]$ is isomorphic to a Lorentz sequence space.*

PROOF. (i) Suppose $[f_n]$ is isomorphic to a subspace X of $d(b,q)$ for some b and $1 \leqq q < +\infty$. By Theorem 24, X contains a complemented subspace which is equivalent to l_q. Hence $1/p + 1/q = 1$. By Proposition 21, $\{f_n\}$ is q-besselian. However, $\{f_n\}$ is equivalent to a symmetric basic sequence in $d(b,q)$ and so by [1, Prop. 5], $\{f_n\}$ is q-hilbertian. Thus $[f_n]$ is isomorphic to l_q, which is a contradiction.

The proof of (ii) is analogous. Q.E.D.

Note that in Corollary 25 we actually prove more; namely, we may replace $\{f_n\}$ by any symmetric basic sequence in $[f_n]$ which is not equivalent to the unit vector basis of l_q.

4.

Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, $1 \leqq p < +\infty$. In this section, we shall give necessary and sufficient conditions that $[f_n]$ has exactly two non-equivalent symmetric basic sequences.

PROPOSITION 26. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, $1 < p < +\infty$, and let $b = a_n^{1/p}$, $n = 1,2,\cdots$. Then*

$$\left\| \sum_{n=1}^{\infty} c_n b_n f_n \right\| \leqq \left(\sum_{n=1}^{\infty} |c_n|^q \right)^{1/q}$$

for all $\{c_n\} \in l_q$ where $1/p + 1/q = 1$.

PROOF. For any $x = \sum_{n=1}^{\infty} \beta_n x_n \in d(a,p)$,

$$\left| \left(\sum_{n=1}^{\infty} c_n b_n f_n \right)(x) \right| \leqq \left(\sum_{n=1}^{\infty} |c_n|^q \right)^{1/q} \left(\sum_{n=1}^{\infty} |\beta_n|^p a_n \right)^{1/p} \leqq \left(\sum_{n=1}^{\infty} |c_n|^q \right)^{1/q} \|x\|.$$

Hence $\| \sum_{n=1}^{\infty} c_n b_n f_n \| \leqq (\sum_{n=1}^{\infty} |c_n|^q)^{1/q}$. Q.E.D.

PROPOSITION 27. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, $1 \leqq p < +\infty$, and let $b_n = a_n^{1/p}$, $n = 1,2,\cdots$. If $g_n = \sum_{i=q_n+1}^{q_{n+1}} \alpha_{i-p_n} f_i$, $n = 1,2,\cdots$, is a block of type* I *of $\{f_n\}$, then*

(i) *when $p = 1$, $\{g_n\}$ is dominated by $\{\sum_{i=p_n+1}^{p_{n+1}} b_{i-p_n} f_i\}$;*

(ii) *when $1 < p < +\infty$, there exists $\{c_n\} \in l_q$, $c_1 \geqq c_2 \geqq \cdots \geqq 0$ such that $\{g_n\}$ is dominated by $\{\sum_{i=p_n+1}^{p_{n+1}} c_{i-p_n} b_{i-p_n} f_i\}$.*

PROOF. (i) Since

$$\sum_{i=p_n+1}^{p_{n+1}} \alpha_i = g_n \left(\sum_{i=p_n+1}^{p_{n+1}} x_i \right) \leqq \left\| \sum_{i=p_n+1}^{p_{n+1}} x_i \right\| \cdot \|g_n\|$$

$$\leqq \left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\| \sum_{i=p_n+1}^{p_{n+1}} a_{n-p_n}, \qquad n = 1,2,\cdots,$$

by Proposition 3, $\{\sum_{i=p_n+1}^{p_{n+1}} b_{i-p_n} f_i\} > \{g_n\}$.

(ii) By [5], there exists $\{c_n\} \in l_q$, $c_1 \geqq c_2 \geqq \cdots \geqq 0$, such that

$$\sum_{i=1}^{n} \alpha_i \leqq 2 \left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\| \sum_{i=1}^{n} c_i b_i, \qquad n = 1,2,\cdots.$$

Again by Proposition 3, $\{\sum_{i=p_n+1}^{p_{n+1}} c_{i-p_n} b_{i-p_n} f_i\} > \{g_n\}$. Q.E.D.

THEOREM 28. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, $1 \leqq p < +\infty$, and let $\{d_n\}$ be the enumeration of the double sequence $\{a_i a_j\}$, $i,j = 1,2,\cdots$ in decreasing order. Let $s_n = \sum_{i=1}^{n} a_i$, $t_n = \sum_{i=1}^{n} d_i$, $n = 1,2,\cdots$, and let*

(i) *every block of type* I *of* $\{f_n\}$ *be equivalent to* $\{f_n\}$;

(ii) $[f_n]$ *have exactly two non-equivalent symmetric basic sequences*;

(iii) $\sup_{1 \leqq n < +\infty} t_n / s_n^{2-1/p} < +\infty$, $1 \leqq p < +\infty$; *and*

(iv) $\sup_{1 \leqq n < +\infty} t_n / s_n < +\infty$.

Then (i) *and* (ii) *are equivalent. Each of the statements* (i) *or* (ii) *implies* (iii). *Furthermore,* (iv) *implies* (*i*). *Thus in the case* $p = 1$, *all the statements are equivalent.*

Proof. (i) $\Rightarrow$ (ii). Let $\{g_n\}$ be a symmetric basic sequence in $[f_n]$. Since $[f_n]$ does not contain any subspace isomorphic to l_1, we may assume that $\{g_n\}$ is a block basic sequence of $\{f_n\}$ and $\| g_n \| = 1$, $n = 1, 2, \cdots$. Let

$$g_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i f_i, \ \alpha_{p_n+1} \geqq \alpha_{p_n+2} \geqq \cdots \geqq \alpha_{p_{n+1}} \geqq 0, \qquad n = 1, 2, \cdots.$$

If $\lim_{n \to \infty} \alpha_n = 0$ then by Lemma 23, $\{g_n\}$ is equivalent to the unit vector basis of l_q when $1 < p < +\infty$ and $1/p + 1/q = 1$, respectively to c_0 when $p = 1$. Otherwise, there exists $c > 0$ such that $\alpha_{p_n+1} \geqq c$, $n = 1, 2, \cdots$. Hence $\{g_n\} > \{f_n\}$. To show $\{f_n\} > \{g_n\}$. Note that if $\sup_{1 \leqq n < +\infty} (p_{n+1} - p_n) < +\infty$ then $\{f_n\} \sim \{g_n\}$. Hence, by taking a subsequence if necessary, we may assume that $p_{n+2} - p_{n+1} > p_{n+1} - p_n$, $n = 1, 2, \cdots$.

Case 1. $p = 1$. Define $f(\sum_{n=1}^{\infty} \beta_n x_n) = \sum_{n=1}^{\infty} \beta_n a_n$ for all $\sum_{n=1}^{\infty} \beta_n x_n \in d(a, 1)$. Then $f \in d(a,1)^*$ and $\| f \| = 1$. Let $h_n = \sum_{i=p_n+1}^{p_{n+1}} a_{i-p_n} f_i$, $n = 1, 2, \cdots$. Then $\{h_n\}$ is a block of type III. By (i) and Lemma 6, $\{h_n\}$ is equivalent to $\{f_n\}$. But $\{h_n\} > \{g_n\}$ by a similar argument used to prove Proposition 27. Hence $\{f_n\} > \{g_n\}$.

Case 2. $1 < p < +\infty$. Let $\gamma_i = \inf_{1 \leqq n < +\infty} \alpha_{p_n+i}$, $i = 1, 2, \cdots$. Then $\gamma_1 \geqq c > 0$ and $\lim_{n \to \infty} \gamma_n = 0$. Suppose there exists $k \in N$ such that $\gamma_{k-1} \neq 0$ and $\gamma_k = 0$. By choosing a subsequence if necessary, we may assume that $\lim_{n \to \infty} \alpha_{p_n+k} = 0$. Let $u_n = \sum_{i=p_n+1}^{p_n+k} \alpha_i f_i$ and $v_n = g_n - u_n$, $n = 1, 2, \cdots$. If $\lim_{n \to \infty} \| v_n \| = 0$ then by choosing a subsequence, we may assume that $\{g_n\} \sim \{u_n\} \sim \{f_n\}$. If $\lim_{n \to \infty} \| v_n \| \neq 0$, then we may assume that $\{v_n\}$ is bounded and the coefficient of $\{v_n\}$ tends to zero. By Lemma 23, and choosing a subsequence if necessary, we may assume that $\{v_n\}$ is equivalent to the unit vector basis of l_q. Hence $\{f_n\} > \{v_n\}$. But $\{f_n\} \sim \{u_n\}$. Thus $\{f_n\} > \{g_n = u_n + v_n\}$. Now it remains to consider the case that $\gamma_n > 0$, $n = 1, 2, \cdots$.

Given an $\varepsilon > 0$, by induction and a standard compactness argument, there exists a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ and

(a) $l_1 < l_2 < \cdots$ in N such that $\gamma_{l_n} < 1/n$, $n = 1,2,\cdots$,

(b) $\{h_n\} \subset d(a,p)^*$ such that $h_n = \sum_{j=1}^{l_n} \beta_j f_j$, $n = 1,2,\cdots$,

and

$$\left\| \sum_{j=1}^{l_n} (\alpha_{p_{n_i}+l_1+\cdots+l_{n-1}+j} - \beta_j) f_{p_{n_i}+l_1+\cdots+l_{n-1}+j} \right\| \leqq (\varepsilon/2^i)(1/n) \qquad i,n = 1,2,\cdots.$$

Let $\alpha = \sum_{n=1}^{\infty} \sum_{j=1}^{l_n} \beta_j f_{l_1+\cdots+l_{n-1}+j} \equiv \sum_{n=1}^{\infty} b_n f_n$. Then $0 \neq \alpha \in d(a,p)^*$. Define $g_n^{(\alpha)} = \sum_{j=p_{n_i}+1}^{p_{n_i+1}} b_{j-p_{n_i}} f_j$. Then $\{f_i\} \sim \{g_i^{(\alpha)}\} \sim \{g_{n_i} - w_i\}$ where

$$w_i = \sum_{j=p_{n_i}+l_1+\cdots+l_i+1}^{p_{n_i+1}} \alpha_j f_j, \qquad i = 1,2,\cdots.$$

However, the coefficient of $\{w_i\}$ tends to zero. Hence either $\{g_{n_i}\} \sim \{g_{n_i} - w_i\}$ or we may assume that $\{w_i\}$ is equivalent to the unit vector basis of l_q. Thus $\{f_i\} \sim \{w_i\}$ and so $\{f_i\} > \{g_{n_i} = g_{n_i} - w_i + w_i\}$.

(ii) $\Rightarrow$ (i). If $[f_n]$ has exactly two non-equivalent symmetric basic sequences then every block of type I of $[f_n]$ is equivalent to $\{f_n\}$. Thus every block of $\{f_n\}$ is equivalent to $\{f_n\}$.

(i) $\Rightarrow$ (iii). If every block of type I of $\{f_n\}$ is equivalent to $\{f_n\}$, by Lemma 4 there exists a constant $K > 0$ such that $\| \sum_{n=1}^{\infty} \beta_n f_n \| \leqq K \| \sum_{n=1}^{\infty} \alpha_n f_n \|^2$ for all $\sum_{n=1}^{\infty} \alpha_n f_n \in [f_n]$ where $\{\beta_n\}$ is any enumeration of $\{\alpha_i \alpha_j\}$, $i,j = 1,2,\cdots$. Given $n \in N$, there exist $n_i \in N$, $i = 1,2,\cdots,k$, such that $n = n_1 + n_2 + \cdots + n_k$, $n_1 \geqq n_2 \geqq \cdots \geqq n_k$, and $t_n = \sum_{i=1}^{k} a_i s_{n_i}$. For $1 < p < +\infty$, let $1/p + 1/q = 1$ and for $p = 1$, let $q = 0$; then

$$\frac{t_n}{s_n^{2-1/p}} = \frac{1}{s_n^{2-1/p}} \left(\sum_{i=1}^{k} a_i \sum_{j=1}^{n_i} a_j f_{n_1+\cdots+n_{i-1}+j} \right) \left(\sum_{i=1}^{n} x_i \right)$$

$$\leqq \frac{K}{s_n^{2-1/p}} \left\| \sum_{i=1}^{n} a_i f_i \right\|^2 \cdot \left\| \sum_{i=1}^{n} x_i \right\| = K \left\| \sum_{i=1}^{n} \left(\frac{a_i}{s_n} \right)^{1/q} a_i^{1/p} f_i \right\|^2 \leqq K \left(\sum_{i=1}^{n} \frac{a_i}{s_n} \right)^{1/q} = K.$$

The last inequality follows from Proposition 26. Hence $\sup_{1 \leqq n < +\infty} t_n / s_n \leqq K$.

(iv) $\Rightarrow$ (i). *Case* 1. $p = 1$. Let $K > 0$ be a constant such that $t_n \leqq K s_n$, $n = 1,2,\cdots$ and let $h_n = \sum_{i=p_n+1}^{p_{n+1}} a_{i-p_n} f_i$, $n = 1,2,\cdots$. By Proposition 27 and the fact that every block of type I of $\{f_n\}$ dominates $\{f_n\}$, it suffices to show that $\{f_n\} > \{h_n\}$.

Suppose $f = \sum_{n=1}^{\infty} \alpha_n f_n$ is convergent. We may assume that $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$ and note that $\|f\| = \sup_{1 \leqq n < +\infty} \sum_{i=1}^{n} \alpha_i / \sum_{i=1}^{n} a_i$ [4]. For any $\sum_{n=1}^{\infty} \beta_n x_n \in d(a,1)$, $\beta_1 \geqq \beta_2 \geqq \cdots \geqq 0$, then

$$\left(\sum_{i=1}^{n} \alpha_i h_i\right)\left(\sum_{i=1}^{\infty} \beta_n x_n\right) = \sum_{i=1}^{n} \alpha_i \sum_{j=p_i+1}^{p_{i+1}} a_{j-p_i}\beta_j \leqq \|f\| \sum_{i=1}^{n} a_i \sum_{j=p_i+1}^{p_{i+1}} a_{j-p_i}\beta_j$$

$$= \|f\| \cdot \left(\sum_{i=1}^{n} a_i h_i\right)\left(\sum_{i=1}^{\infty} \beta_n x_n\right).$$

Hence $\| \sum_{i=1}^{n} \alpha_i h_i \| \leqq \|f\| \cdot \| \sum_{i=1}^{n} a_i h_i \| \leqq \|f\| \sup_n t_n/s_n \leqq K\|f\|$. Thus $\{f_n\} > \{h_n\}$.

Case 2. $1 < p < +\infty$. By Lemma 4, it suffices to show that if $f = \sum_{n=1}^{\infty} \alpha_n f_n$, $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$, is convergent then $\sum_{n=1}^{\infty} \gamma_n f_n$ is convergent where $\{\gamma_n\}$ is the enumeration of $\{\alpha_i \alpha_j\}$, $i,j = 1,2,\cdots$. By [5],

$$\left\|\sum_{n=1}^{\infty} \alpha_n f_n\right\| = \inf_{\{c_n\}\in M_n} \sup_{1\leqq n<+\infty} \sum_{i=1}^{n} \alpha_i \Big/ \sum_{i=1}^{n} c_i b_i$$

where $b_n = a_n^{1/p}$, $n = 1,2,\cdots$, and

$$M_q = \left\{\{c_n\} \in l_q : c_1 \geqq c_2 \geqq \cdots \geqq 0, \left(\sum_{n=1}^{\infty} c_n^q\right)^{1/q} \leqq 1\right\}.$$

Let $\{c_n\} \in M_q$ such that $\sum_{i=1}^{n} \alpha_i \leqq 2\| \sum_{n=1}^{\infty} \alpha_n f_n \| \sum_{i=1}^{n} c_i b_i$, $n = 1, 2, \cdots$. Let $\{\delta_n\}$ be the enumeration of $\{c_i b_i c_j b_j\}$, $i,j = 1,2,\cdots$ in decreasing order. Then $\sum_{i=1}^{n} \gamma_i \leqq 2\|f\| \sum_{i=1}^{n} \delta_i$. To show that $\sum_{i=1}^{\infty} \gamma_n f_n$ is convergent, by Proposition 3, it remains to show that $\sum_{n=1}^{\infty} \delta_n f_n$ is convergent. Let $\sum_{n=1}^{\infty} \beta_n x_n \in d(a,p)$, $\beta_1 \geqq \beta_2 \geqq \cdots \geqq 0$. Then

$$\left(\sum_{n=1}^{\infty} \delta_n f_n\right)\left(\sum_{n=1}^{\infty} \beta_n x_n\right) \leqq (\sum |c_i c_j|^q)^{1/q} (\sum |b_i b_j|^p \beta_n^p)^{1/p} \leqq \left(\sum_{n=1}^{\infty} \beta_n d_n\right)^{1/p}$$

$$\leqq K^{1/p} \left(\sum_{n=1}^{\infty} \beta_n^p a_n\right)^{1/p} = K^{1/p} \left(\sum_{n=1}^{\infty} \beta_n x_n\right)$$

where $K = \sup_{1\leqq n<+\infty} t_n/s_n$. Hence $\sum_{n=1}^{\infty} \delta_n f_n$ is convergent. This completes the proof of the theorem. Q.E.D.

REMARK 29. In the proof of Theorem 28, (i) ⇒ (ii), Case 2, for $1 < p < +\infty$, the proof actually includes the case $p = 1$. We give the proof for $p = 1$ here because of its simplicity.

Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$. We now study the symmetric basic sequences in $[f_n]$ which span a complemented subspace of $[f_n]$.

PROPOSITION 30. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, $1 < p < +\infty$, and let $1/p + 1/q = 1$. If $\sum_{n=1}^{\infty} \alpha_n f_n$ is convergent then for any $p_1 < p_2 < \cdots$ in N, $\sum_{n=1}^{\infty} \| \sum_{i=p_n+1}^{p_{n+1}} \alpha_i f_i \|^q < +\infty$.*

PROOF. For each $n = 1,2,\cdots$, let $y_n = \sum_{i=p_n+1}^{p_{n+1}} \beta_i x_i$ such that $\| y_n \| = 1$ and $\sum_{i=p_n+1}^{p_{n+1}} \alpha_i \beta_i = \| \sum_{i=p_n+1}^{p_{n+1}} \alpha_i f_i \|$. Since $\{y_n\}$ is a bounded block basic sequence of $\{x_n\}$ in $d(a,p)$, $\{y_n\}$ is p-Hilbertian [1, Prop. 5]. Thus $\sum_{n=1}^{\infty} c_n y_n$ is convergent for any $\{c_n\} \in l_p$. Hence $(\sum_{n=1}^{\infty} \alpha_n f_n)(\sum_{n=1}^{\infty} c_n y_n) = \sum_{n=1}^{\infty} c_n \| \sum_{i=p_n+1}^{p_{n+1}} \alpha_i f_i \|$ is convergent. This implies that $\{\| \sum_{i=p_n+1}^{p_{n+1}} \alpha_i f_i \|\} \in l_q$. Q.E.D.

LEMMA 31. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, $1 < p < +\infty$, and let $1/p + 1/q = 1$. If $\{g_n\}$ is a block basic sequence of $\{f_n\}$ which is equivalent to the unit vector basis of l_q, then $[g_n]$ is complemented in $d(a,p)^*$.*

PROOF. Let $g_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i f_i$, $n = 1,2,\cdots$. We may assume that $\| g_n \| = 1$, $n = 1,2,\cdots$. Let $y_n = \sum_{i=p_n+1}^{p_{n+1}} \beta_i x_i$ such that $\| y_n \| = 1$ and $\sum_{i=p_n+1}^{p_{n+1}} \alpha_i \beta_i = \| g_n \| = 1$, $n = 1,2,\cdots$. For any $\sum_{n=1}^{\infty} \gamma_n f_n \in d(a,p)^*$, by Proposition 30, $\{\| \sum_{i=p_n+1}^{p_{n+1}} \gamma_i f_i \|\} \in l_q$. Hence $\sum_{i=1}^{\infty} \| \sum_{i=p_n+1}^{p_{n+1}} \gamma_i f_i \| g_n$ is convergent. Thus $\sum_{n=1}^{\infty} (\sum_{i=p_n+1}^{p_{n+1}} \gamma_i \beta_i) g_n$ is convergent. Define $P(\sum_{n=1}^{\infty} \gamma_n f_n) = \sum_{n=1}^{\infty} (\sum_{i=p_n+1}^{p_{n+1}} \gamma_i \beta_i) g_n$. Then P is well defined and it is easy to see that P is linear and $P(g_n) = g_n$, $n = 1,2,\cdots$. By the uniform boundedness principle, it is clear that P is bounded. Q.E.D.

THEOREM 32. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, $1 \leqq p < +\infty$. Then every block of type I of $\{f_n\}$ is equivalent to $\{f_n\}$ if and only if for every symmetric block basic sequence $\{g_n\}$ of $\{f_n\}$, $[g_n]$ is complemented in $[f_n]$.*

PROOF. If every block of type I of $\{f_n\}$ is equivalent to $[f_n]$, by Theorem 28, $[f_n]$ has exactly two non-equivalent symmetric basic sequences. Let $\{g_n\}$ be a symmetric block basic sequence of $\{f_n\}$. If $[g_n]$ is isomorphic to l_q then by Lemma 31, $[g_n]$ is complemented when $1 < p < +\infty$. In the case $p = 1$, then $[g_n]$ is isomorphic to c_0. Since $[f_n]$ is separable, so $[g_n]$ is complemented in $[f_n]$. Now if $\{g_n\}$ is equivalent to $\{f_n\}$, by Proposition 13, we may assume that $g_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i f_i$, $\alpha_{p_n+1} \geqq \alpha_{p_n+2} \geqq \cdots \geqq \alpha_{p_{n+1}} \geqq 0$, $n = 1,2,\cdots$. By Lemma 23, there exists $c > 0$ such that $\alpha_{p_n+1} \geqq c$, $n = 1,2,\cdots$. Define

$$P\left(\sum_{n=1}^{\infty} \beta_n f_n\right) = \sum_{n=1}^{\infty} \frac{\beta_{p_n+1}}{\alpha_{p_n+1}} g_n \text{ for all } \sum_{n=1}^{\infty} \beta_n f_n \in [f_n].$$

It is easy to see that P is a projection onto $[g_n]$.

Conversely, if every symmetric block $\{g_n\}$ of type I of $\{f_n\}$ spans a complemented subspace in $[f_n]$, then, by Theorem 8, $\{g_n\} \sim \{f_n\}$. By the argument given in Corollary 12, every block of type I of $\{f_n\}$ is equivalent to $\{f_n\}$. Q.E.D.

COROLLARY 33. *There exists a Banach space X with symmetric basis $\{x_n\}$*

such that for every symmetric block basic sequence $\{y_n\}$ of $\{x_n\}$, $[y_n]$ is complemented in X but X is not isomorphic to c_0 or l_p, $1 \leqq p < +\infty$.

PROOF. Let $a_1 = a_2 = 1$, $a_n = 1/\log n$, $n = 3,4,\cdots$, and let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$, $1 \leqq p < +\infty$. Then $\{f_n\}$ is a symmetric basis of $X \equiv [f_n]$. We will show that $\sup_{1 \leqq n < +\infty} \sum_{i=1}^n d_i / \sum_{i=1}^n a_i < +\infty$ where $\{d_n\}$ is the enumeration of $\{a_i a_j\}$, $i,j = 1,2,\cdots$ in decreasing order. Then, by Theorem 28, every block of type I of $\{f_n\}$ is equivalent to $\{f_n\}$. Hence, by Theorem 32, every symmetric block basic sequence of $\{f_n\}$ spans a complemented subspace in X. Let $b_1 = b_2 = 1$, $b_n = \log(n-1)/(\log n)^2$, $n = 3,4,\cdots$. Then it is easy to see that $\sum_{i=1}^n a_i \sim \sum_{i=1}^n b_i \sim n/\log n$. Now, for each $n \in N$, there exist $n_1 \geqq n_2 \geqq \cdots \geqq n_k$ in N such that $n = n_1 + n_2 + \cdots + n_k$ and

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{k} a_i s_{n_i} \sim \sum_{i=1}^{k} a_i a_{n_i} n_i.$$

Note that $k \leqq n_1$ and $a_i a_{n_k+1} \leqq a_1 a_{n_1}$, $i = 1,2,\cdots,k$, and $a_{n+1} \geqq \frac{1}{2} a_n$, $n = 1,2,\cdots$. Then $\log n \leqq \log(kn_1) \leqq 2\log n_1$. Hence $\sum_{i=1}^n d_i / \sum_{i=1}^n a_i \sim \sum_{i=1}^k a_i a_{n_i} n_i / \sum_{i=1}^n a_i \sim \sum_{i=1}^k a_i a_{n_i} n_i / (n/\log n) \leqq 2\log n_1 / n \sum_{i=1}^k 2a_i a_{n_1+1} n_i \leqq 4(\log n_1/n)\, a_1 a_{n_1} \sum_{i=1}^k n_i = 4$. Thus

$$\sup_{1 \leqq n < +\infty} \sum_{i=1}^{n} d_i \Big/ \sum_{i=1}^{n} a_i < +\infty. \qquad \text{Q.E.D.}$$

REMARK 34. By a result of J. Lindenstrauss and T. Tzafriri [6], a Banach space X with unconditional basis $\{x_n\}$ is isomorphic to either c_0 or l_p, $1 \leqq p < +\infty$, if and only if for every permutation π of N and every block basis $\{y_k\}$ of $\{x_{\pi(n)}\}$ there exists a projection in X whose range is the subspace generated by $\{y_k\}$. Hence if $\{x_n\}$ is a symmetric basis of a Banach space X, then X is isomorphic to either c_0 or l_p, $1 \leqq p < \infty$, if and only if every block basic sequence of $\{x_n\}$ spans a complemented subspace in X.

REMARK 35. Using the argument in Theorem 19, we can prove the following result. Let $\{x_n, f_n\}$ be the unit vector basis in $d(a,p)$, $1 \leqq p < +\infty$. Then $[f_n]$ has exactly two non-equivalent symmetric basic sequences if and only if for every symmetric basic sequence $\{g_n\}$ in $[f_n]$ there exists a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ such that $[g_{n_i}]$ is complemented in $[f_n]$.

Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,1)$. Lemma 36 yields the surprising result that $d(a,1)$ and $[f_n]$ cannot simultaneously have exactly two non-equivalent symmetric basic sequences. Recall that $d(a,1)$ has exactly two non-equivalent symmetric basic sequences if and only $\sup_{1 \leqq n,k < +\infty} s_{nk}/s_n s_k < +\infty$ where $s_n = \sum_{i=1}^n a_i$, $n = 1,2,\cdots$ [1, Th. 6]. We need the following lemma.

LEMMA 36. *Let $d(a,p)$, $1 \leqq p < +\infty$, be a Lorentz sequence space. Then $0 < \inf_{1 \leqq n\,k < +\infty} s_{nk}/s_n s_k \leqq \sup_{1 \leqq n,k < +\infty} s_{nk}/s_n s_k < +\infty$ if and only if there exists $1 < q < +\infty$ such that $d(a,p)$ is isomorphic to $d(b,p)$ where $b_n = 1/n^{1/q}$, $n = 1,2,\cdots$.*

PROOF. Let $t_n = \sum_{i=1}^n b_i$, $n = 1,2,\cdots$, where $b_n = n^{-1/q}$, $n = 1,2,\cdots$, and $1 < q < +\infty$. Let $1/q + 1/q' = 1$. Then $t_n \sim n^{1/q'}$. Hence $0 < \inf_{1 \leqq n,k < +\infty} t_{nk}/t_n t_k \leqq \sup_{1 \leqq n,k < +\infty} t_{nk}/t_n t_k < +\infty$. If $d(a,p)$ is isomorphic to $d(b,p)$ then $s_n \sim t_n$ [1, Lem. 2]. Hence $0 < \inf_{1 \leqq n,k < +\infty} s_{nk}/s_n s_k \leqq \sup_{1 \leqq n,k < +\infty} s_{nk}/s_n s_k < +\infty$. Conversely, let $M > 0$ such that $1/M \leqq s_{nk}/s_n s_k \leqq M$, $n,k = 1,2,\cdots$. Then $(1/M^k) s_{n^k} \leqq s_n^k \leqq M^k s_{n^k}$ for all n,k. Thus there exists a constant $0 \leqq c \leqq 1$ such that $s_n \sim n^c$ (see for example, [11, p. 614–615]). Since $d(a,p)$ is not isomorphic to c_0, we have $c \neq 0$. Also, since $\lim_{n\to\infty} s_n/n = 0$, it follows that $c \neq 1$. Let $q' = 1/c$ and $1/q + 1/q' = 1$. Then $s_n \sim n^{1/q'} \sim t_n$ where $t_n = \sum_{i=1}^n b_i$ and $b_n = n^{-1/q}$, $n = 1,2,\cdots$. Thus $d(a,p)$ is isomorphic to $d(b,p)$. Q.E.D.

THEOREM 37. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,1)$. If every block of type I of $\{f_n\}$ is equivalent to $\{f_n\}$ then $0 < \inf_{1 \leqq n,k < +\infty} s_{nk}/s_n s_k \leqq \sup_{1 \leqq n\,k < +\infty} s_{nk}/s_n s_k = +\infty$.*

PROOF. By Theorem 28, $\sup_{1 \leqq n < +\infty} \sum_{i=1}^n b_i / \sum_{i=1}^n a_i < +\infty$ where $\{b_n\}$ is the rearrangement of $\{a_i a_j\}$, $i,j = 1,2,\cdots$ in decreasing order. Hence for any $n,k = 1,2,\cdots$, $s_n s_k \leqq \sum_{i=1}^{nk} b_i$. Thus $\sup_{1 \leqq n,k < +\infty} s_n s_k/s_{nk} \leqq \sup_{1 \leqq n,k < +\infty} \sum_{i=1}^{nk} b_i / \sum_{i=1}^{nk} a_i < +\infty$. That is, $\inf_{1 \leqq n,k < +\infty} s_{nk}/s_n s_k > 0$. Now suppose $\sup_{1 \leqq n,k < +\infty} s_{nk}/s_n s_k < +\infty$. By Lemma 36, we may assume that $a_n = n^{-1/q}$, $n = 1,2,\cdots$, for some $1 < q < +\infty$. It remains to show that in this case, $\sup_{1 \leqq n < +\infty} \sum_{i=1}^n b_i / \sum_{i=1}^n a_i = +\infty$ where $\{b_n\}$ is the enumeration of $\{a_i a_j\}$, $i,j = 1,2,\cdots$ in decreasing order.

Let $1/q + 1/q' = 1$. For each $n \in N$, let $m = n!$ and $m_k = m/k$, $k = 1,2,\cdots,n$. Then

$$\sum_{i=1}^{m_1+\cdots+m_n} b_i \geqq \sum_{i=1}^{n} a_i s_{m_i} \sim \sum_{i=1}^{n} a_i m_i^{1/q'} = \sum_{i=1}^{n} m^{1/q'}/i \sim m^{1/q'} \log n$$

and

$$\sum_{i=1}^{m_1+\cdots+m_n} a_i \sim (m_1 + \cdots + m_n)^{1/q'} \sim (m \log n)^{1/q'}.$$

Hence

$$\sum_{i=1}^{m_1+\dots+m_n} b_i \Big/ \sum_{i=1}^{m_1+\dots+m_n} a_i \geqq (\log n)^{1/q}$$

and so $\sup_{1 \leqq n < +\infty} \sum_{i=1}^{n} b_i / \sum_{i=1}^{n} a_i = +\infty$. Q.E.D.

5.

Motivated by Corollary 33 and Remark 34, in this section we study a class of Banach spaces X with unconditional basis $\{x_n\}$ such that every bounded block basic sequence of $\{x_n\}$ spans a complemented subspace in X.

THEOREM 38. *Let E be a Banach space with unconditional basis $\{u_n\}$ such that for every bounded block basic sequence $\{y_n\}$ of $\{u_n\}$, $[y_n]$ is complemented in E. For any strictly increasing sequence $\{p_n\}$ in N, let $X_n = [u_{p_n+1}, u_{p_n+2}, \cdots, u_{p_{n+1}}]$ in E, $n = 1,2,\cdots$, and let $X = (\sum_{n=1}^{\infty} \oplus X_n)_{l_p}$, $1 \leqq p < +\infty$, (or $(\sum_{n=1}^{\infty} \oplus X_n)_{c_0}$). If $x_1 = (u_1,0,0,\cdots)$, $x_2 = (0,u_2,0,\cdots)$, $x_3 = (0,u_3,0,\cdots),\cdots$ is the natural basis in X then every bounded block basic sequence of $\{x_n\}$ spans a complemented subspace of X.*

PROOF. For each n, let $E_n = [x_{p_n+1}, x_{p_n+2}, \cdots, x_{p_{n+1}}]$. Let

$$y_n = \sum_{i=q_n+1}^{q_{n+1}} \alpha_i x_i, \qquad n = 1,2,\cdots,$$

be a bounded block basic sequence of $\{x_n\}$. Let $\{y_{n_i}\}$ be the subsequence of $\{y_n\}$ consisting of all y_n with the properties that $y_{n_i} \in E_k$ for some $k \in N$. Define $z_{n_i} = \sum_{j=q_{n_i}+1}^{q_{n_i+1}} \alpha_j u_j$, $i = 1,2,\cdots$. Then $\{z_{n_i}\}$ is a bounded block basic sequence of $\{u_n\}$. Let P_0 be a projection from E onto $[z_{n_i}]$. For each $n = 1,2,\cdots$, let P_n be the restriction of P_0 on E_n. Then $\sup_{1 \leqq n < +\infty} \| P_n \| \leqq \| P_0 \|$. Thus there exists a projection P from X onto $[y_{n_i}]$ (see, for example, [11, p. 542]). Since $\{x_n\}$ is an unconditional basis in X, we may assume that the unconditional basis constant of $\{x_n\}$ is 1. Hence the projection Q on X defined by $Q(x_j) = x_j$ if $q_{n_i} + 1 \leqq j \leqq q_{n_i+1}$ for some $i \in N$ and $Q(x_j) = 0$ otherwise is of norm one. Let $P_1 = PQ$. Then P_1 is a projection from X onto $[y_{n_i}]$ such that $P_1(x_j) = 0$ if

$$x_j \notin \{x_{q_{n_i}+1}, \cdots, x_{q_{n_i+1}}\}, \ i = 1,2,\cdots.$$

Now let $\{y_{k_j}\}$ be the subsequence of $\{y_n\}$ consisting of all the y_n which are not in $\{y_{n_i}\}$. Note that if $\{x_{q_{k_{2j_0}}+1}, \cdots, x_{q_{k_{2j_0}+1}}\} \wedge E_n \neq \emptyset$ for some $n \in N$ then

$$\{x_{q_{k_{2j}}+1}, \cdots, x_{q_{k_{2j}+1}}\} \wedge E_n = \emptyset$$

for all $j \neq j_0$. Hence

$$\left\| \sum_{j=1}^{\infty} \sum_{i=q_{k_{2j}}+1}^{q_{k_{2j+1}}} \alpha_i x_i \right\| = \left(\sum_{j=1}^{\infty} \left\| \sum_{i=q_{k_{2j}}+1}^{q_{k_{2j+1}}} \alpha_i x_i \right\|^p \right)^{1/p}.$$

Let $F_{2j} = [x_{q_{k_{2j}}+1}, \cdots, x_{q_{k_{2j+1}}}]$ $j = 1, 2, \cdots$. For each $j \in N$, let $g_j \in F_{2j}^*$ such that $g_j(y_{k_{2j}}) = 1$ and $\| g_j \| = 1 / \| y_{k_{2j}} \|$. Define

$$P_2 \left(\sum_{n=1}^{\infty} a_n x_n \right) = \sum_{j=1}^{\infty} g_j \left(\sum_{i=q_{k_{2j}}+1}^{q_{k_{2j+1}}} a_i x_i \right) y_{k_{2j}}$$

for all $\sum_{n=1}^{\infty} a_n x_n \in X$. P_2 is clearly linear and $P_2(y_{k_{2j}}) = y_{k_{2j}}, j = 1, 2, \cdots$. Now

$$\left\| P_2 \left(\sum_{n=1}^{\infty} a_n x_n \right) \right\| = \left\| \sum_{j=1}^{\infty} g_j \left(\sum_{i=q_{k_{2j}}+1}^{q_{k_{2j+1}}} a_i x_i \right) y_{k_{2j}} \right\| \leqq \left\| \sum_{j=1}^{\infty} \| g_j \| \left\| \sum_{i=q_{k_{2j}}+1}^{q_{k_{2j+1}}} a_i x_i \right\| y_{k_{2j}} \right\|$$

$$= \left(\sum_{j=1}^{\infty} \| g_j \|^p \left\| \sum_{i=q_{k_{2j}}+1}^{q_{k_{2j+1}}} a_i x_i \right\|^p \| y_{k_{2j}} \|^p \right)^{1/p}$$

$$= \left(\sum_{j=1}^{\infty} \left\| \sum_{i=q_{k_{2j}}+1}^{q_{k_{2j+1}}} a_i x_i \right\|^p \right)^{1/p} = \left\| \sum_{j=1}^{\infty} \sum_{i=q_{k_{2j}}+1}^{q_{k_{2j+1}}} a_i x_i \right\|$$

$$\leqq \left\| \sum_{j=1}^{\infty} a_j x_j \right\|.$$

Hence P_2 is a bounded projection from X onto $[y_{k_{2j}}]$. Similarly, there exists a projection P_3 from X onto $[y_{k_{2j-1}}]$. It is easy to see that $P_1 + P_2 + P_3$ is a projection from X onto $[y_n]$. Q.E.D.

REMARK 39. When $p_n = \frac{1}{2}n(n+1)$, $n = 1, 2, \cdots$, and $E = l_p$, $1 < p \neq 2 < +\infty$, A. Pelczynski [10] has shown that $\{x_n\}$ is an unconditional but not symmetric basis of X.

COROLLARY 40. *In* l_p, $1 < p \neq 2 < +\infty$, *there exists an unconditional basis* $\{x_n\}$ *which is non-symmetric and such that every bounded block basis sequence of* $\{x_n\}$ *spans a complemented subspace in* l_p.

COROLLARY 41. *There exists a Banach space X with unconditional basis* $\{x_n\}$ *such that every bounded block basic sequence of* $\{x_n\}$ *spans a complemented subspace in X and X is not isomorphic either to* c_0 *or* l_p, $1 \leqq p < +\infty$.

PROOF. Let $E = l_2$ and $\{u_n\}$ be the natural basis in l_2. For $p_n = \frac{1}{2}n(n+1)$ $n = 1, 2, \cdots$, let $X = (\sum_{n=1}^{\infty} \oplus X_n)_{l_1}$ and $\{x_n\}$ be the natural basis in X as in Theorem 38. Then the Banach space X with unconditional basis $\{x_n\}$ has the required properties. Q.E.D

References

1. Z. Altshuler, P. G. Casazza and B. L. Lin, *On symmetric basic sequences in Lorentz sequence spaces*, Israel J. Math. (to appear).

2. C. Bessaga and A. Pelczynski, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–174.

3. P. G. Casazza and B. L. Lin, *On conditional bases in Banach spaces*, Rev. Roumaine Math. Pures Appl. (to appear).

4. D. J. H. Garling, *On symmetric sequence spaces*, Proc. London Math. Soc. (3) **16** (1966), 85–105.

5. D. J. H. Garling, *A class of reflexive symmetric BK-spaces*, Canad. J. Math. **21** (1969), 602–608.

6. J. Lindenstrauss and L. Tzafriri, *On the complemented subspaces problem*, Israel J. Math. **9** (1971), 263–269.

7. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math. **10** (1971), 379–390.

8. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces II*, Israel J. Math. **11** (1972), 355–379.

9. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces III* (to appear).

10. A. Pelczynski, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–228.

11. I. Singer, *Bases in Banach spaces I*, Springer-Verlag, 1970.

12. A. E. Tong, *Diagonal submatrices of matrix maps*, Pacific J. Math. **32** (1970), 555–559.

13. L. Tzafriri, *An isomorphic characterization of L_p and c_0-spaces II*, Michigan Math. J. **18** (1971), 21–31.

Department of Mathematics
The University of Alabama
Huntsville, Alabama, U. S. A.

and

Department of Mathematics,
The University of Iowa
Iowa City, Iowa, U. S. A.